Determining interval lengths for fuzzy time series forecasting model based on index of fuzzy sets by combining hedge algebra and particle swarm optimization

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Received 10 Sep. 2023; Revised 5 Nov. 2023; Accepted 15 Nov. 2023; Published 10 Dec. 2023
DOI: https://doi.org/10.54939/1859-1043.j.mst.FEE.2023.271-282

ABSTRACT

Researchers frequently use fuzzy time series (FTS) forecasting models to estimate future values since they do not rely on the same rigid assumptions as traditional forecasting techniques. There are generally four factors that determine the performance of the FTS forecasting model (1) determining the length of intervals in the universe of discourse, (2) fuzzification rules or feature representation of crisp time series, (3) establishing fuzzy relation groups (FRGs) and (4) creating defuzzification rule to get crisp forecasted value. Considering the first factor and the fourth factor, we propose the hybrid FTS forecasting model combining particle swarm optimization (PSO) and hedge algebra (HA) to improve forecasting accuracy. Where the hedge algebra is utilized as a tool for partitioning the universe of discourse into intervals of different lengths. Then, the times series data are fuzzified into fuzzy sets, the fuzzy relationship groups are established and forecasting output value based on the index of fuzzy sets is calculated. Ultimately, the suggested model collaborates with PSO to obtain the optimal intervals determined by HA. To test the proposed model, we conduct a simulated study on two widely used real-time series and compare the performance with some recently developed models. Error statistics, such as MSE and RMSE show that the proposed model performs better than the comparing models.

Keywords: Enrolments; Fuzzy time series; FRGs; Hedge algebras; PSO.

1. INTRODUCTION

In the past few years, numerous studies on forecasting challenges have been presented in an effort to improve forecasting accuracy and reduce computation time. Due to this, Song and Chissom introduced two FTS forecasting models [1, 2] in 1993. These models were based on Zadeh's fuzzy set theory [3], which provided a theoretical framework to represent a particular dynamic process whose observations are linguistic values. Their two FTS forecasting models, which have a forecasting structure that consists of five fundamental phases, have been used to predict enrollments at universities in Alabama: (1) determining the universe of discourse (UoD), (2) partitioning the UoD into intervals, (3) defining the fuzzy sets and fuzzifying the time series, (4) establishing fuzzy logical relationships, and (5) forecasting and defuzing the forecasting value. In comparison to conventional statistical models such as regression analysis, moving average and autoregressive moving average, two forecasting models in recent investigations [2, 3] may yield better prediction outcomes with forecasting challenges with linguistic values or uncertainty. Nonetheless, when the fuzzy rule matrix is huge as well as lacks persuasiveness in specifying the length of intervals, their models require a long time to compute and, as a consequence. To overcome these drawbacks, Chen [4] constructed a novel forecasting model based on fuzzy relationship groups and simple arithmetic operations in the defuzzification process. Furthermore, research studies [5, 6] emphasized the significance of giving weights to address the matter of recurrent fuzzy relationships and to indicate the difference in their relevance in the FTS model. Additionally, the relevance of the high-order FTS model [7] and the influence of interval lengths [8] on the one-factor and two-factor FTS models [9] provides the basis for the strong growth of FTS models in the coming time periods. Chen and Tanuwijaya [10] applied the automated clustering approach to split the UoD into different interval lengths during the forecasting model's fuzzification step. Other studies [11- 15] utilize optimization approaches with various FTS models to adapt and discover the lengths of intervals from the UoD. Research work in [16] proposed a linguistic time series forecasting model based on hedge algebra quantification [17] to convert numerical time series into linguistic for anticipating university enrollments. Furthermore, the authors [18] presented a linguistic time series forecasting model based on the linguistic forecasting rules instead of the linguistic relationship groups. However, two of these models are only concerned with developing a first-order forecasting model to predict the number of enrollments at the University of Alabama.

In this study, we have developed a hybrid FTS forecasting model that incorporates both HA and PSO techniques to address a range of challenges. The linguistic terms represent data values within the fuzzy time series and are quantified using HA, enabling the segmentation of the UoD into unevenly sized periods. After obtaining these intervals, the time series dataset is fuzzified utilizing predefined fuzzy sets, resulting in fuzzy logic relationships (FLRs) that are subsequently classified [15].

2. BASIC DESCRIPTIONS AND PRELIMINARIES

In this section, the basic review of fuzzy time series [1, 2], the hedge algebras[17] and PSO algorithm [19] are briefly summarized.

2.1. Basic definitions related to FTS

This section briefly reviews the definitions in the forecasting models of Song and Chissom [1, 2] as definitions are also improved by authors in research works [7]

Definition 1. Let U be the UoD, where $U = \{u_1, u_2, \dots, u_n\}$. A fuzzy set A_i of Ucan be defined as follows:

$$A_i = \frac{\mu_{A_i}(u_1)}{u_1} +, \dots, + \frac{\mu_{A_i}(u_2)}{u_2}, \dots, + \frac{\mu_{A_i}(u_n)}{u_n}$$

Where μ_{A_i}: $U \rightarrow [0,1]$ is the membership function of A_i , $\mu_{A_i}(u_i)$ indicates the degree of membership of u_i in the fuzzy set A, $\mu_{A_i}(u_i) \epsilon [0, 1]$ and $1 \leq i \leq n$.

Definition 2. Fuzzy time series [1, 2]

Let $Y(t)$ $(t = 0, 1, 2, \dots)$, a subset of real numbers, be the UoD on which the fuzzy sets $f_i(t)$ (i $= 1,2 \dots$) are defined. $F(t)$ is a collection of $f_1(t), f_2(t), \dots f_i(t), \dots$, then $F(t)$ is called a fuzzy time series defined on $Y(t)$.

Definition 3. λ- order fuzzy logical relationship [7]

Let $F(t)$ be a fuzzy time series. If $F(t)$ is caused by $F(t-1), F(t-2), \dots, F(t-\lambda+1), F(t-\lambda)$ then this fuzzy logical relationship is represented by $F(t-\lambda), \dots, F(t-2), F(t-1) \rightarrow F(t)$ and is called an λ- order fuzzy time series. Here, the symbol λ is called the order of fuzzy logical relationship.

2.2. Some basic concepts of hedge algebras

Nguyen Cat Ho introduces hedge algebras in the paper [17]. This theory is viewed as a novel method for quantifying linguistic concepts in the context of time series forecasting, in contrast to the fuzzy set method. The HA principles are used in this work as a foundation to divide the discourse of time series into beginning intervals of varying durations. Assume that there is a collection of linguistic values for the variable X, and that they are sorted in the manner described below:

$\mathcal{X} = \{Very\ Very\ low < Very\ low < low < Little\ low < Very\ Little\ low < medium < Very\ Little\ high < Little\ big < high < Very\ high < \cdots\}$.

Each linguistic variable $\mathcal{X}$ is represented by an algebraic structure as $\mathcal{AX} = (X, G, C, H, \leq)$ and called HA, where X is the set of terms in $\mathcal{X}$; $\leq$ denotes a natural semantically ordering relation on X; $G = \{c^-, c^+\}$, $c^- \leq c^+$, is the set of primary generators, in which c^+ and c^- are, respectively, the negative primary term and the positive one of a linguistic variable $\mathcal{X}$, C = $\{0, 1, w\}$ a set of constants, with $(0 \leq c^- \leq W \leq c^+ \leq 1)$; $H = H^- \cup H^+$ with $H^- = \{h_{-q} \geq \cdots \geq h_{-2} \geq h_{-1}\}$ is the set of all negative hedges of X, $\forall h \in H^-$ then $hc^+ \leq c^+$ and $H^+ = \{h_1 \leq h_2 \leq \cdots \leq h_p\}$ is the set of all positive ones of X, $\forall h \in H^+$ then $hc^+ \geq c^+$.

Definition 4. Let $\mathcal{AX}$ = (*X*, *G*, *C*, *H*, ≤) be a HA. A function fm: $X \rightarrow [0, 1]$ is said to be a fuzziness measure of terms in *X* if:

1). $fm(c^-) + fm(c^+) = 1$ and $\sum_{h \in H} fm(hx) = fm(x)$ for $\forall x \in X$

2). For the constants 0, W and 1, $fm(0) = fm(W) = fm(1) = 0$

3). For $\forall x, y \in X, \forall h \in H$, $\frac{fm(hx)}{fm(x)} = \frac{fm(hy)}{fm(y)}$, that is this proportion does not depend on specific elements and it is called the fuzziness measure of the hedge *h* and is denoted by $\mu(h)$.

2.3. Particle swarm optimization algorithm

Kennedy and Eberhart [19] first proposed particle swarm optimization, which imitates birds flying to discover food sources. The fundamental PSO is described briefly here:

Assume S is a set of potential solutions represented by a swarm of particles: $S = \{x_{1d}, x_{2d}, \ldots, x_{Nd}\}$, where each particle x_i was moved through the search space (*d*-dimensional space) to search the optimal solution having its position Y_{id}^t at iteration *t* computed as:

$$Y_{id}^{t+1} = Y_{id}^t + V_{id}^{t+1} \tag{1}$$

where V_{id}^{t+1} is the velocity of particle x_i at cycle *t* + 1, which is computed as:

$$V_{id}^{t+1} = \omega^t * V_{id}^t + c_1 * r1() * \left(P_{best\_id}^t - Y_{id}^t\right) + c_2 * r2() * \left(G_{best} - Y_{id}^t\right) \tag{2}$$

where G_{best}^t and $P_{best\_id}^t$ are the global and local solutions that are found up to cycle *t*, respectively; c_1 and c_2 are self-cognitive and social cognitive factors; r_1 and r_2 are two random numbers which uniformly distributed in [0, 1]; ω^t is inertia weight which is calculated according the equation (3).

$$\omega^t = \omega_{max} - \frac{t * (\omega_{max} - \omega_{min})}{iter\_max} \tag{3}$$

where $iter\_max$ is the maximum iteration number.

3. A FTS FORECASTING MODEL USING HA AND PSO

This section introduces a hybrid FTS-based forecasting model that combines both HA and PSO techniques, comprising three stages depicted in figure 1. The following is a description of the three phases of the suggested model.

3.1. The proposed forecasting model based on FTS and HA

In this subsection, an FTS forecasting model that uses hedge algebras to forecast university enrollments is presented. By quantitatively mapping language concepts into fuzzy intervals based on HA. On each of the acquired intervals, we define fuzzy sets and fuzzy historical data. The FLRs and fuzzy relationship groups were established using the fuzzified rule in accordance with [15]. The proposed model can be presented in the following manner, step by step:

Step 1. Define the universe of discourse *U* of historical time series data

Let $U = [D_{min} - D_1, D_{max} + D_2]$ is the *UoD*. To define U, the minimum value D_{min} and the maximum value D_{max} of the historical time series data is defined. From historical enrolments time series, U is defined as U = [13000, 20000], where D_{min} = 13055, D_{max} = 19337, D_1 = 55, D_2 = 663, LU = 7000.

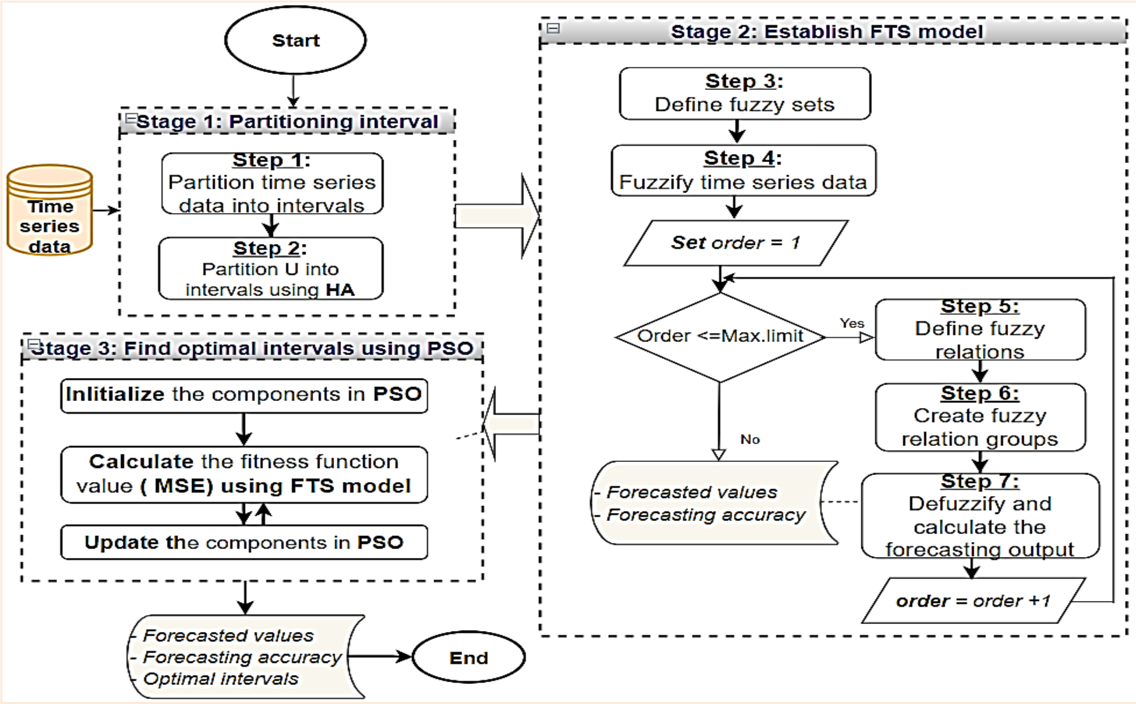

Figure 1. *The flowchart of the proposed forecasting model.*

Step 2. Divide U into different intervals based on HA

This step uses HA with structure as $\mathcal{AX} = (X, G, C, H, \leq)$, where X is the set of terms of the linguistic variable "*enrollments*"{ X= dom(*enrollments*)}; $\leq$ denotes a natural semantically ordering relation on X; $G = \{c^-, c^+\}$ ={*Low, High}, Low (Lw)* ≤ *High* (*Hi*); $C = \{0, w, 1\}$ a set of constants, with $(0 \leq c^- \leq W \leq c^+ \leq 1)$ and H ={ *Very, Little* }. To compare the forecasted results of the proposed model with other models. This study uses the number of intervals equal to 7 and 14, which are the number of linguistic terms used to quantify the enrollments' time series values. Based on the given linguistic terms, we identified the corresponding intervals for them in the universe of discourse U.

Step 2.1. The domain U = [13000, 20000] is mapped to the domain [0, 1]

Assume that the value of 16807 in the time series dataset is the low value, then the fuzziness measure of terms is calculated as follows: $fm(low) = \frac{16807-13000}{20000-13000}$ = 0.54, $fm(high)$ = 1- 0.54 = 0.46 and LU = 20000 -13000 = 7000. Mapping these values to U, we have covfm(low) and covfm(high) that are determined, respectively as $fm(low)$ x LU = 0.54 x 7000 = 3780, $fm(high)$ x LU = 0.46 x 7000 = 3220. In this paper, we can choose $\mu(Little)$ = 0.4, $\mu(Very)$ = 1- 0.4 = 0.6. Based on $\mu(Little), \mu(Very)$ value, the value of α, β is determined as 0.4, 0.6, respectively.

Step 2.2. Define the fuzzy interval of linguistic variables in the universe of discourse

Based on step 2.1, The linguistic values of terms belonging to the fuzziness interval are calculated as follows:

$covfm(A_1)$ = μ($Verry$) x μ($Very$) x $covfm$ (Low) = 0.6*0.6*3780 = 1361;

--;

$covfm(A_7)$ = μ($Very$) x $covfm(High)$ = 0.6*3220 = 1932

The intervals corresponding to linguistic terms are obtained by mapping the value of linguistic terms to the domain U as follows:

For seven linguistic terms, obtaining seven intervals as u_1= [13000, 14361), u_2= [14361, 15268.2) , u_3= [15268.2, 15873), u_4= [15873, 16808), u_5= [16808, 17605), u_6= [17605, 18068), u_7= [18068, 20000]

For 14 linguistic terms, obtaining 14 intervals as u_1= [13000, 14361), u_2= [14361, 14723.9) , u_3= [14723.9, 15268.2), u_4= [15268.2, 15631),…, u_{14}= [18840.8, 20000].

Step 3. Define linguistic terms A_i which is represented by fuzzy sets

Each interval in step 2 represents a linguistic value of the linguistic variable "enrolments". For seven intervals, there are seven linguistic values to represent different regions in the universe of discourse on *U*. Each linguistic value represents a fuzzy set A_i and its definitions is described in formulas (4) and (5) as follows.

$$A_i = {a_{i1}}/{\mathrm{u}_1} + {a_{i2}}/{\mathrm{u}_2} + \ldots + {a_{ij}}/{\mathrm{u}_j} + \cdots + {a_{i7}}/{\mathrm{u}_7} \quad (4)$$

$$a_{ij} = \begin{cases} 1\ j = i \\ 0.5\ j = i-1\ or\ j = i+1 \\ 0\ otherwise \end{cases} \quad (5)$$

Here, the symbol '+' denotes the set union operator, a_{ij} ∈[0,1] ($1 \le i \le 7$, $1 \le j \le 7$), u_j is the jth interval of the universe of discourse. The value of a_{ij} indicates the grade of membership of u_j in the fuzzy set A_i.

Step 4. Fuzzy the historical time series data

To fuzzify the historical time series data, it is essential to obtain the degree of membership value of each data value belonging to each u_i for each year. If the maximum membership value of one day's observation occurs at u_i , and ($1 \le i \le 7$), then the fuzzified value for that particular year is considered as A_i.

Step 5. Create the λ– order fuzzy logical relationships (λ ≥ 1)

Based on Definition 3, to establish a λ-order fuzzy logical relationship, we should find out any relationship which has the $F(t-\lambda), F(t-\lambda+1), \ldots, F(t-1) \rightarrow F(t)$, where $F(t-\lambda), F(t-\lambda+1), \ldots, F(t-1)$ and $F(t)$ are called the current state and the next state, respectively. Then, a λ - order fuzzy relationship in the training phase is obtained by replacing the corresponding linguistic values.

Step 6. Establish all time-variant fuzzy relationship groups (TV-FRGs)

In earlier research studies [4, 5], recurrent fuzzy logic relationships were either disregarded when establishing fuzzy relationship groups or if mentioned, they were not considered in terms of chronological order. In the current study, we adopt the concept of time-variant fuzzy relationship groups[15] to construct Fuzzy Relationship Groups, which we refer to as TV-FRGs.

Step 7. Defuzzify and calculate the forecasting output values

To defuzzify the fuzzified data values, the defuzzified principle in equation(6) is presented to compute the forecasted value for all first–order and high–order TV- FRGs in the training phase.

Support that the fuzzified data of year *t* -1 is $\mathrm{A_j}$ and there is the fuzzy relationship group whose current state is $\mathrm{A_j}$, shown as follows: $A_j(t-1) \rightarrow A_{i1}(t1), A_{i2}(t2), A_{ip}(tk)$. The forecasting value output of time *t* defined as follows:

$$Forecasted_{value} = \frac{index\_A_{i1} \times m_{i1} + index\_A_{i2} \times m_{i2} + \cdots + index\_A_{ip} \times m_{ip}}{\sum_{k=1}^{p} index\_Aik} \quad (6)$$

Where m_{i1}, m_{i2}, m_{ip} are the middle values of the intervals u_{i1} , u_{i2} and u_{ip} respectively, $index\_A_{i1}, index\_A_{i2}, \ldots, index\_A_{ip}$ are index of k^{th} fuzzy set in the TV-FRG and p denotes the total number of fuzzy sets on the next state of TV-FRG.

Then, we use a defuzzified principle [13] for computing with the unknown linguistic value in the testing phase.

The efficiency of the proposed model is evaluated using various statistical indexes, namely Mean Square Error (MSE) and Root Mean Square Error (RMSE). The evaluation criteria are determined by the following equations:

$$MSE = \frac{1}{n}\sum_{i=\lambda}^{n}(F_i - R_i)^2 \quad (7)$$

$$\text{RMSE} = \sqrt{\frac{1}{n}\sum_{i=\lambda}^{n}(F_i - R_i)^2} \quad (8)$$

Where, R_i and F_i note the actual and forecasted value at time *i*, respectively, *n* is the total number of years to be forecasted, λ is the order of fuzzy logical relationship.

3.2. The hybrid FTS forecasting model combining HA and PSO

This section introduces the hybrid FTS model that merges HA and PSO techniques. Specifically, PSO is employed to minimize the MSE (7) by adjusting the lengths of the initial intervals determined by HA. A brief explanation of the proposed model is given as follows. For the training phase, each particle is used to represent the partitioning of time series data (e.g., n intervals). Assume that the lower bound and upper bound of the universe of discourse U be y_0 and y_n , respectively. Each particle denotes a vector containing $n-1$ elements as y_1, y_2, …, y_{n-2} and y_{n-1}, where $(1 \le i \le n-1)$ and $y_i \le y_{i+1}$. From these $n-1$ elements, define the n intervals as $u_1 = [y_0, y_1]$, $u_2 = [y_1, y_2]$,,..., $u_i = [y_{i-1}, y_i]$,… and $u_n = [y_{n-1}, y_n]$, respectively. During the training phase, the proposed model allows each particle to move from its present location to other positions by (1) and (2), and then repeats the steps until the stopping condition is met. If the stopping requirement is fulfilled, then all of the FRGs acquired by the global best position (G_{best}) among all personal best positions (P_{best}) of all particles used to predict the new testing data throughout the testing phase are employed. The following are the entire steps of the proposed model, as shown in Algorithm 2:

Algorithm 2. The training phase algorithm of the proposed model

1**. Input**: Historical time series data

2**. Output**: The forecasting results and the MSE value (MSE = *Gbest* = min(*Pbest*))

Begin

3. Define the initial intervals by applying HA and **use** the forecasting steps in Subsection 3.1 to reach the initial forecasting accuracy (MSE).

4. Initialize: a population of N particles

- ✓ The initial position X_{id} and the velocity V_{id} of all particles, respectively.
- ✓ The initial personal best position vectors of the i^{th} particle are the same as its initial position vector at the beginning and find *Gbest*

5. **do**

5.1**. foreach** all particle *id*, ($1 \le id \le N$) **do**

- ✓ Following the steps in Subsection 3.1 sequentially, from step 3 to step 7, such as defining linguistic terms, fuzzify all historical, determining all $\lambda-$order fuzzy logical relationships, establishing all $\lambda-$order TV- FRGs, defuzzify forecasting values, calculating the MSE values for particle *id*
- ✓ The new *Pbest* of particle id is saved according to the MSE values.

end for

5.2. The new *Gbest* of all particles is saved according to the MSE values
6. **foreach** all particle id, ($1 \leq id \leq N$) **do**
- ✓ The particle *id* is moved to another position according to Eqs. (1) and (2)

end for
- ✓ Update ω according to Eq. (3)

while (the maximum moving steps(*iter_max*) or the minimum MSE are reached)
End.

4. EXPERIMENTAL RESULTS AND DISCUSSIONS

In this research, we have used the proposed model to forecast the number of enrollments at the University of Alabama [3], the datasets of Gas prices RON95 in Vietnam which was collected from https://vnexpress.net/kinhdoanh/hang-hoa and the time series data of "killed" in car road accidents in Belgium [20]. The proposed model has executed 20 independent runs with different amounts of orders and intervals using the PSO parameters used for each dataset, which are provided in table 1 to produce forecasting results. Then, the best result of all runs is produced for reporting and comparing with other forecasting models.

Table 1. *The parameters of PSO are used in the proposed model for different time series data.*

The parameters of PSO	For enrolments	For RON95	For car road accident
Number of particles *N*	30	30	30
Max number of iterations	150	150	150
Value of inertial weight ω	ω_{max}= 0.9 to ω_{min} = 0.4	ω_{max}= 0.9 to ω_{min} = 0.4	ω_{max}= 0.9 to ω_{min} = 0.4
Coefficient c1 = *c2*	2	2	2
The limited range of *V*	[-100, 100]	[-100, 100]	[-50, 50]
The limited range of *X*	[13000, 20000]	[21700, 24240]	[953, 1644]

4.1. Experimental results for forecasting enrolments

4.1.1. Experimental results based on the first–order FTS

In order to evaluate the effectiveness of the suggested model, which is based on first-order FTS with intervals of 7, forecasting models from articles [16, 18, 21-24] were compared. The parameters are taken from the statistics on enrollments. The proposed model receives the lowest RMSE value of 189.5 among all the compared models, as shown in table 2. Differences between the proposed model and the models mentioned above are the way in which the fuzzy relationship group and the method of partitioning the universe of discourse.

Table 2. *A comparison of the forecasting results of the proposed model with its counterparts based on first–order FTS under seven intervals.*

Year	Real data	[21]	[22]	[23]	[16]	[24]	[18]	Proposed model
1972	13563	13486	13944	14279	14537	13500	13742	13618.4
---	---	---	---	---	---	---	---	---
1992	18876	18808	18933	19257	19217	18855	19129	19060.3
RMSE		**578.3**	**506**	**445.2**	**512.18**	**350.9**	**325.9**	**189.5**

Additionally, the suggested forecasting model is assessed against other models [4, 6, 11, 13-15, 23, 25] utilizing 14 intervals. The anticipated outcomes are presented in table 3. It is apparent that among the examined forecasting models, the proposed model achieves the lowest MSE value of 5023.4.

Table 3. *A comparison of the forecasting results of the proposed model with its counterparts based on first–order FTS under a number of intervals of 14.*

Year	Real data	[4]	[25]	[6]	[11]	[13]	[14]	[15]	[23]	Proposed model
1971	13055									
1972	13563	14000	13584	13653	13714	13555	13579	13434	13512	13528.6
----	----	----	----	----	----	----	----	----	-----	----
1992	18876	19000	19084	19059	19014	19014	19031	18820	18718	18870.6
MSE		**407507**	**65413**	**31684**	**35324**	**22965**	**8224**	**7475**	**14534**	**5023.4**

4.1.2. Experimental results based on the high–order forecasting model

All historical datasets [3] from the years 1971 to 1992 are divided into two portions in order to compare the proposed model with the present methods based on various high-orders. The training data set is made up of the first component's 19 observations, which cover the period from 1971 to 1989, while the testing data set is made up of the second component's three observations.

Case (1): Experimental results in the training phase

In this stage, the forecasting accuracies cited from studies [7, 12-14, 15] are chosen for comparison with the proposed model based on a number of intervals equal to 7. With the number of different orders, For easy visualization, figure 2 depicts the trend in terms of forecasting accuracy between the proposed model and its counterparts. Among all FLRS orders done, the proposed model obtains the lowest MSE value of 180.03 with 8th-order fuzzy relationships. From these curves, it can be seen that the forecasting accuracy of the proposed model is more precise than those of compared models under different high-order FLRs at all.

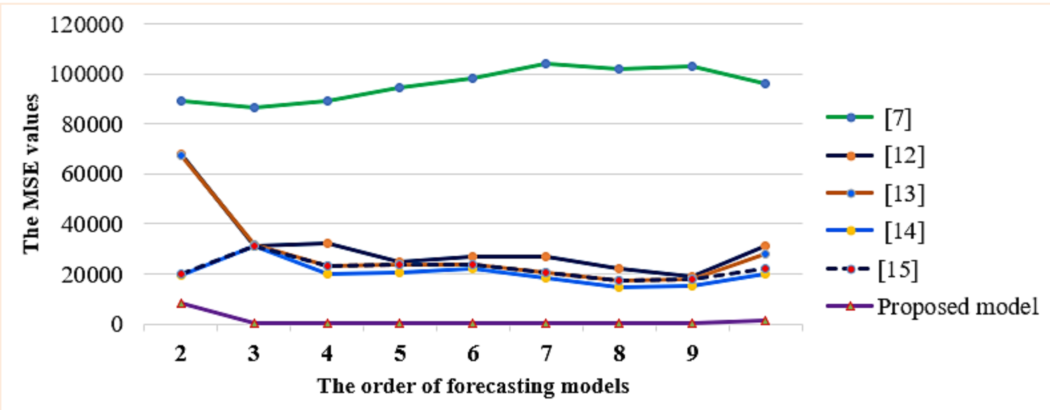

Figure 2. *The graphs compare the MSE values of the proposed model with those of its competitors that are based on a number of high-order FLRs.*

Furthermore, our proposed model has been compared with similar models discussed in previous works [7, 12-14, 26]. In table 4, the proposed model and its counterparts were evaluated in terms of forecasting accuracy, considering different high-orders and the number of intervals equal to 14. The results in table 6 clearly demonstrate that our proposed method achieves the lowest MSE value when compared to the other models listed. This highlights the superior performance of our approach in accurately forecasting enrollments when compared to the methods presented in the table.

Table 4. *A comparison of the forecasting results obtained between the proposed model and its counterparts based on the various high-order FTS with 14 intervals.*

Models	[7]	[26]	[12]	[13]	[14]	Proposed model
MSE	**86694**	**53084**	**1101**	**234**	**173**	**16.5**

In summary, the aforementioned results demonstrated that the proposed model outperforms the current models based on the high-order FTS with various numbers of intervals for forecasting enrolments in the training phase.

Case (2): Experimental results in the testing phase

For the testing phase, we can only predict the new enrollment for the following year based on enrollment data from previous years. For example, enrollment data from 1971 to 1989 is used to forecast new enrollment for 1990. Similarly, based on enrollments from 1971 to 1990, a new enrolment for 1991 may be forecasted. After the proposed has thoroughly trained the training data, future data values can be calculated and compared to those of the forecasting models [4, 11, 14, 26]. The forecasted results of the proposed model based on the third-order FLR with a different number of intervals and the highest vote W_h=15 (constant-defined by the user) is compared with other forecasting models and shown in table 5. From the obtained results in these tables, it can be seen that the proposed model obtains the lowest RMSEs value among five compared models, respectively.

Table 5. *A comparison of the forecasting results between the proposed model and other models with the number of intervals = 7 and which use vote W_h=15.*

Models	[4]	[11]	[14]	ATVF-KM[26]	ATVF-PSO[26]	Proposed model
RMSE	**829.9**	**836.5**	**612.8**	**160.23**	**107.12**	**79.27**

4.2. Experimental results for forecasting the "killed" in car road accidents

In addition, the proposed model is also used for forecasting vehicle road accidents in Belgium [20] from 1974 to 2004 and making a comparison of the forecasted results with the previous research works [20, 27-30]. A comparison of the forecasted results using RMSE is shown in table 6. More detailed comparison, at the same number of intervals of 13, the proposed model obtains the smallest RMSE value of 1.67 among two models [27, 28] using the 3rd-order FTS and also has a far smaller RMSE value than models [20, 29, 30] based on first - order FTS with different number of intervals.

Table 6. *A comparison of the forecasting results of the proposed model with different models for forecasting car road accidents.*

Models	[27]	[20]	[30]	[29]	[28]	Proposed model	
						First order	***Third order***
RMSE	**46.78**	**37.66**	**41.61**	**32.0**	**19.2**	**15.06**	**1.67**

4.3. Experimental results for forecasting Gas price RON95 in Vietnam

In this section, we apply the proposed model to forecast the gas price of Vietnam with all historical data for the period from 01/01/2023 to 21/06/2023. We implemented the proposed model under the second-order and kept a number of intervals equal to 14, which is obtained by using HA. The performance of the proposed model is evaluated by using the MSE (7). The forecasted results of the proposed model are presented in table 7 and also depicted graphically in figure 3.

Table 7. *The forecasted results of the proposed model for the Gas price RON95.*

Date	Actual RON95	Forecasted values
11/01/2023	22150	22161
---	---	---
21/06/2023	22010	22019
22/06/2023	N/A	22310
MSE	**3805.3**	

From figure 3, it can be seen that the forecasted trend of the proposed model is close to the actual data based on the high-order FTS. To sum up, the demonstrations above show that the proposed model outperforms the existing models based on high-order FTS model with various numbers of intervals for forecasting the different problems.

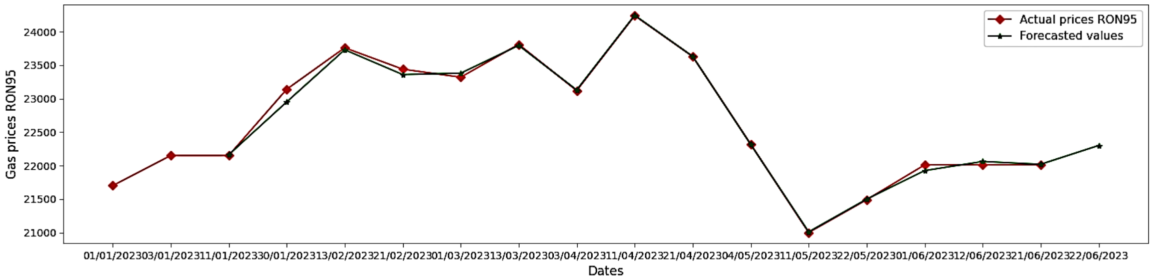

Figure 3. *The curves depict the trend actual values and forecasted values based on second-order FLRs.*

5. CONCLUSIONS

This study expands the hybrid FTS forecasting model, which combines PSO with hedging algebras, to forecast a variety of issues. Two aspects that are thought to be crucial and have a significant impact on how accurate forecasting models are optimizing the duration of intervals and constructing fuzzy relationship groupings. To get over the drawbacks of FTS models utilizing fuzzy connection groups, the suggested model utilizes the idea of time-variant fuzzy relationship groups to compute forecasting output results. Prior to now, determining interval lengths was typically done using soft computing techniques like evolutionary processes or clustering algorithms. In the suggested forecasting model, we produce intervals of varying lengths by employing a HA-based mathematical structure. In addition, the PSO optimization technique is applied in optimizing the lengths of intervals from the universe of discourse with a view to improving the accuracy of the forecasting model. The forecasting effectiveness of the suggested model has been greatly increased by combining HA and PSO techniques. The simulated results on the datasets of Gas price RON95 in Vietnam showed that the proposed model gives noticeably better forecasting in case of the high-order FTS, and the compared results on the dataset of the University of Alabama and the dataset of the killed-in-car road accidents show that, in many cases, the proposed model gets better forecasting performance than the existing ones. Details of the comparison are shown in tables 2 -7.

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TÓM TẮT

Xác định độ dài khoảng cho mô hình dự báo chuỗi thời gian mờ dựa trên chỉ số tập mờ sử dụng đại số gia tử và tối ưu bày đàn

Các nhà nghiên cứu thường sử dụng các mô hình dự báo chuỗi thời gian mờ (FTS) để ước tính các giá trị trong tương lai vì chúng không phụ thuộc vào những giả định nghiêm ngặt như các kỹ thuật dự báo truyền thống. Thông thường, có bốn yếu tố quyết định đến hiệu quả của mô hình dự báo FTS (1) xác định độ dài khoảng tập nền, (2) quy tắc mờ hóa hoặc biểu diễn đặc điểm của chuỗi thời gian rõ, (3) thiết lập các nhóm quan hệ mờ (FRGs) và (4) tạo quy tắc giải mờ để nhân được các giá trị đầu ra rõ. Xem xét yếu tố đầu tiên và yếu tố thứ tư, chúng tôi đề xuất một mô hình dự báo chuỗi thời gian mờ sử dụng tối ưu bày đàn (PSO) và đại số gia tử (HA) để cải thiện độ chính xác dự báo. Trong đó, đại số hedge được sử dụng như một công cụ để chia tạp nền thành các khoảng có độ dài khác nhau. Sau đó, dữ liệu chuỗi thời gian được mờ hóa thành các tập mờ, các nhóm quan hệ mờ được thiết lập và giá trị dự báo dựa trên chỉ số của các tập mờ được tính toán. Cuối cùng, mô hình đề xuất được kết hợp với PSO để đạt được các khoảng tối ưu đã xác định bởi HA. Để đánh giá mô hình đề xuất, chúng tôi tiến hành nghiên cứu mô phỏng trên hai chuỗi thời gian thực được sử dụng rộng rãi và so sánh hiệu suất với một số mô hình được phát triển gần đây. Thống kê sai số dự báo bằng MSE và RMSE cho thấy mô hình đề xuất hoạt động tốt hơn các mô hình so sánh.

Từ khóa: Tuyển sinh; Chuỗi thời gian mờ; Nhóm quan hệ mờ; Đại số gia tư; Tối ưu bày đàn.