The multiple-set split feasibility problem and its application in intensity-modulated radiation therapy

Nguyen Minh Trang*

Faculty of International Training, Thai Nguyen University of Technology.
*Corresponding author: nguyenminhtrang@tnut.edu.vn
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ABSTRACT

In this paper, we apply a parallel iterative method for solving the split common fixed point problem in Hilbert spaces, without prior knowledge of the transfer operator norm, to the multiple-set split feasibility problem, which plays an important role in medical image reconstruction and signal processing. By formulating a model of the intensity-modulated radiation therapy treatment in a mathematical framework based on the multiple-set split feasibility problem, we also offer an effective approach for solving the intensity-modulated radiation therapy treatment planning.

Keywords: Split common null point problem; Multiple-set split feasibility problem; Intensity-modulated radiation therapy.

1. INTRODUCTION

Let C and Q be nonempty, closed, and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. The split feasibility problem (SFP) is formulated as follows:

$$\text{Find an element } x^* \in S = C \cap A^{-1}(Q).$$

This problem was first introduced in 1994 by Censor and Elfving with the aim of modeling certain inverse problems.

A more general version of the SFP is the multiple-set split feasibility problem (MSFP), which requires finding a point closest to a family of closed convex sets in one space such that its image under a linear transformation will be closest to another family of closed convex sets in the image space, that is,

$$\text{Find an element } x^* \in \left(\bigcap_{i=1}^{N} C_i\right) \cap A^{-1}\left(\bigcap_{j=1}^{M} Q_j\right) \quad \text{(I)}$$

where C_i, $i = 1, 2, \ldots, N$, and Q_j, $j = 1, 2, \ldots, M$, are nonempty, closed and convex subsets of H_1 and H_2, respectively.

The SFP has drawn many authors' attention since it arises in many fields in the real world, such as signal processing, image reconstruction, and medical care. See, for instance, the discretized model of the SFP in the image reconstruction problem of X-ray tomography by Qu and Liu [6], or a model of the intensity-modulated radiation therapy by Censor et al. [3].

A popular algorithm for solving the SFP is Byrne's CQ method [1]: for any starting point $x_0 \in H_1$, one defines the sequence $\{x_n\}$ by

$$x_{n+1} = P_C^{H_1}\left[x_n - \gamma A^*\left(I^{H_2} - P_Q^{H_2}\right)Ax_n\right]$$

for all $n \geq 0$, where $P_C^{H_1}$ and $P_Q^{H_2}$ are the metric projections from H_1 onto C and H_2 onto Q, respectively, $A^* : H_2 \to H_1$ is the adjoint of A, and $\gamma \in \left(0, \frac{2}{\|A\|^2}\right)$.

Many iterative methods for solving the SFP and the MSFP based on Byrne's CQ algorithm

involve step sizes that depend on the norm of the transfer operator *A*. We know that the computation of the operator norm is generally not easy in practice.

An alternative approach to solve the MSFP is through the split common fixed point problem (SCFPP) since the MSFP is a special case of the SCFPP. Let $T_i : H_1 \to H_1,\ i = 1, 2, \ldots, N$, and $S_j : H_2 \to H_2, j = 1, 2, \ldots, M$, be two finite families of mappings, and let $A : H_1 \to H_2$ be a bounded linear operator, then the SCFPP is the problem that

$$\text{Find an element } x^* \in \Omega := \left(\bigcap_{i=1}^{N} Fix(T_i)\right) \cap A^{-1}\left(\bigcap_{j=1}^{M} Fix(S_j)\right) \tag{II}$$

In this paper, an algorithm for solving the SCFPP is introduced in section 2, in which the step size is selected in such a way that the implementation of the algorithm does not require any prior information regarding the norm of the operator. In section 3, we then apply the algorithm to solve the MSFP. Finally, in section 4, we present a model of the intensity-modulated radiation therapy treatment in a mathematical framework based on the MSFP.

2. PRELIMINARIES

The SCFPP for two nonexpansive mappings T_1 and S_1 was first introduced in 2009 by Censor and Segal [4], they also propose a weak convergence algorithm,

$$x_{n+1} = T_1\left(x_n - \rho A^*\left(I^{H_2} - S_1\right)Ax_n\right)$$

where $\rho \in \left(0, \frac{2}{\|A\|^2}\right)$.

In 2014, Cui and Wang [5] proposed an algorithm in which the step size does not depend on the norm of the transfer operator *A*, specifically

$$x_{n+1} = \tilde{T}_1\left(x_n - \rho_n A^*\left(I^{H_2} - S_1\right)Ax_n\right)$$

where $\tilde{T}_1 = (1-\lambda)I^{H_1} + \lambda T_1,\ \lambda \in (0,1)$ and

$$\rho_n = \begin{cases} \dfrac{(1-c)Tx_n - S_1(Tx_n)^2}{2T^*\left(Tx_n - S_1(Tx_n)\right)^2}, & Tx_n \neq S_1(Tx_n), \\ 0, \quad \text{otherwise}, \end{cases} \tag{1}$$

with $c < 1$. Cui and Wang proved the weak convergence of the algorithm with suitable conditions on λ and *c*.

Also, using the step size ρ_n determined by (1), in 2015, Boikanyo [2] improved the algorithm to achieve strong convergence, for $u \in H_1$,

$$x_{n+1} = \alpha_n u + (1-\alpha_n)\tilde{T}_1\left(x_n - \rho_n A^*\left(I^{H_2} - S_1\right)Ax_n\right), \quad n \geq 0.$$

For $\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty}\alpha_n = 0$, and $\sum_{n=1}^{\infty}\alpha_n = \infty$, they proved the sequence $\{x_n\}$ converges strongly to $P_\Omega u$.

Generally, such algorithms achieve only weak convergence. Several strong convergence algorithms were then proposed for the split common fixed point problem for two nonexpansive mappings, but not many similar studies have been conducted for the general problem with two finite families of nonexpansive mappings, problem (II).

In this section, using the viscosity approximation method, as well as a modification of the CQ method, we propose a new algorithm for solving problem (II), in which the step size does not depend on the norm of the transfer operator.

Algorithm 2.1. [7] For any $u, x_0 \in H_1$, let $\{x_n\}$ be the sequence generated by

$y_{i,n} = \tilde{T}_i x_n, i = 1, 2, ..., N,$
choose i_n such that $||y_{i_n,n} - x_n|| = \max_{i=1,2,...,N}\{||y_{i,n} - x_n||\}$ and let $y_n = y_{i_n,n}$,
$z_{j,n} = S_j(Ay_n), j = 1, 2, ..., M,$
choose j_n such that $||z_{j_n,n} - Ay_n|| = \max_{i=1,2,...,N}\{||z_{j,n} - Ay_n||\}$ and let $z_n = z_{j_n,n}$,
$t_n = y_n + \delta_n A^*(z_n - Ay_n),$
$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)t_n, n \geq 0,$

where $\tilde{T}_i = \beta_{i,n} I + (1-\beta_{i,n})T_i, i = 1,2,...,N$; $\{\beta_{i,n}\}, \{\alpha_n\}$, $i = 1,2,...,N, n \geq 0$, are sequences of positive real numbers, $f : H_1 \to H_1$ is a strict contraction mapping from H_1 into itself with the contraction coefficient $c \in [0,1)$ and

$$\delta_n = \begin{cases} \rho_n \dfrac{||z_n - Ay_n||^2}{||A^*(z_n - Ay_n)||^2}, ||A^*(z_n - Ay_n)|| > 0, \\ 0, \text{otherwise}, \end{cases}$$

where $\{\rho_n\} \subset [\alpha, \beta] \subset (0,1)$.

Theorem 2.2. [7] Let $T_i : H_1 \to H_1, i = 1, 2, ..., N$, and $S_j : H_2 \to H_2, j = 1, 2, ..., M$, be two finite families of mappings, and let $A : H_1 \to H_2$ be a bounded linear operator such that

$$\Omega := \left(\bigcap_{i=1}^{N} Fix(T_i)\right) \cap A^{-1}\left(\bigcap_{j=1}^{M} Fix(S_j)\right) \neq \varnothing.$$

If the conditions

C1) $\{\beta_{i,n}\} \subset [c,d] \subset (0,1)$ for all $i = 1,2,\ldots,N$;

C2) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

are satisfied, then the sequence $\{x_n\}$ generated by algorithm 2.1 converges strongly to $x^\dagger \in \Omega$, which is the unique solution to the variational inequality

$$\langle (I^{H_1} - f)x^\dagger, y - x^\dagger \rangle \geq 0 \ \ \forall y \in \Omega.$$

3. ALGORITHM FOR MSFP

Fixed point problems for nonexpansive mappings have practical importance in mathematics and physics because many problems in these fields can be cast in the form of such fixed point problems. For instance, the well-known split feasibility problem becomes a common split fixed point problem where the nonexpansive mappings are the metric projections onto the relevant closed and convex sets. This is one of the reasons why algorithms for finding fixed points of nonexpansive mappings have been of great interest.

For nonempty, closed, and convex subsets C_i, i = 1, 2, …,N, and Q_j, j = 1, 2, …,M, of Hilbert spaces H_1 and H_2, respectively; let $T_i = P_{C_i}^{H_1}$ and $S_j = P_{Q_j}^{H_2}$ where $P_{C_i}^{H_1}, P_{Q_j}^{H_2}$ are the metric projections from H_1 onto C_i and H_2 onto Q_j, respectively, we have

$$C_i = Fix\left(P_{C_i}^{H_1}\right) = Fix(T_i), Q_j = Fix\left(P_{Q_j}^{H_2}\right) = Fix(S_j)$$

Thus, problem (I) then becomes problem (II). Using theorems 2.2, we arrive at the following result regarding the solution of the MSFP in Hilbert spaces.

Corollary 3.1. Let C_i, i = 1, 2, …,N, and Q_j, j = 1, 2, …,M, are nonempty, closed and convex

subsets of H_1 and H_2, respectively, and let $A : H_1 \to H_2$ be a bounded linear operator such that

$$\Omega := \left(\bigcap_{i=1}^{N} C_i\right) \cap A^{-1}\left(\bigcap_{j=1}^{M} Q_j\right) \neq \varnothing$$

If the conditions C1) and C2) are satisfied, then the sequence $\{x_n\}$ generated by algorithm 2.1 with $T_i = P_{C_i}^{H_1}, i = 1, 2, \ldots, N,$ and $S_j = P_{Q_j}^{H_2}, j = 1, 2, \ldots, M,$ converges strongly to a point $x^* \in \Omega$ which is the unique solution to the variational inequality.

$$\langle (I^{H_1} - f)x^*, y - x^* \rangle \geq 0 \ \ \forall y \in \Omega.$$

4. MSFP IN INTENSITY-MODULATED RADIATION THERAPY

While the MSFP is potentially useful for a variety of inversion problems that can be formulated, we describe briefly, in this section, a specific application in the Intensity Modulated Radiation Therapy (IMRT) that is motivated by [3].

IMRT is a type of radiation treatment for cancer. Specifically, IMRT is a form of external beam radiation therapy. With external beam radiation therapy, a machine called a linear accelerator directs radiation in the form of high-energy X-ray beams toward cancer cells, destroying them. IMRT uses sophisticated technology to design energy beams that vary in strength or intensity, allowing oncologists to direct radiation to cancer cells while minimizing the exposure of nearby healthy tissue to harmful amounts of radiation.

Let us first define the notation. We divide the entire volume of the patient into I voxels, enumerated by $i = 1, 2, \ldots, I$. Assume that T anatomical structures have been outlined, including planning target volumes (PTVs) and the organs at risk (OAR). We denote the set of voxels indices in structure t by S_t. Individual voxels i may belong to several sets S_t, i.e., different structures may overlap. We will further assume that the radiation is delivered independently from each of the J beamlets, which are arranged in a certain geometry and indexed by $j = 1,2,\ldots, J$. The intensities x_j of the beamlets are arranged in a J-dimensional vector $x = (x_j)_{j=1}^{J} \in R^J$, in the J-dimensional Euclidean space R^J- the radiation intensity space.

The quantities $d_{ij} \geq 0$, which represents the dose absorbed in voxel i due to radiation of unit intensity from the jth beamlet, are calculable by any forward calculation program. Let h_i denote the total dose absorbed in voxel i and let $h = (h_i)_{i=1}^{I}$ be the vector of doses absorbed in all voxels. We call the space R^I the dose space.

We can now calculate h_i as

$$h_i = \sum_{j=1}^{J} d_{ij} x_j \tag{2}$$

The *dose influence matrix* $D = (d_{ij})$ is the $I \times J$ matrix whose elements are the d_{ij}'s mentioned above. Thus, (2) can be rewritten as the vector equation

$$h = Dx \tag{3}$$

Finally, let us assume that we have M constraints in the dose space and N constraints in the intensity space. Let H_m be the set of dose vectors that fulfil the mth dose constraint, and let X_n be the set of beamlet intensity vectors that fulfil the nth intensity constraint. Let us consider some concrete examples of the sets H_m and X_n. Each of the constraint sets H_m and X_n can be one of the specific H and X sets, respectively, described below.

In the dose space, a typical constraint is that, in a given critical structure S_t , the dose should not exceed an upper bound u_t. The corresponding set $H_{\max,t}$ is

$$H_{\max,t} = \{h \in R^I \mid h_i \leqslant u_t, \text{ for all } i \in S_t\}.$$

Similarly, in the target volumes (TVs), the dose should not fall below a lower bound l_t . The set $H_{\min,t}$ of dose vectors that fulfil this constraint is

$$H_{\min,t} = \{h \in R^I \mid l_t \leqslant h_i, \text{ for all } i \in S_t\}.$$

To handle EUD constraints for each volume of interest S_t , consisting of N_t voxels, a real-valued function $E_t : R^I \to R$, called the EUD function, is defined as

$$E_t(h) = \left(\frac{1}{N_t} \sum_{i \in S_t} (h_i)^{\alpha_t} \right)^{1/\alpha_t} \tag{4}$$

The parameter α_t is a tissue-specific number which is negative for target volumes and positive for OAR. For α_t = 1, the EUD function is the mean dose of the organ for which it is calculated. On the other hand, letting $\alpha_t \to \infty$ makes the EUD function approach the maximal value: $\max\{h_i \mid i \in S_t\}$. The EUD constraint for an upper EUD bound e_t for a structure S_t can be described by the set

$$H_{\text{EUD},t} = \{h \in R^I \mid E_t(h) \leqslant e_t\}.$$

Lower EUD bounds can be described similarly.

Due to the non-negativity of the dose, $h \geq 0$, the EUD function of (4) is convex for all $\alpha_t \geq 1$ and concave for all $\alpha_t \leq 1$. Therefore, the constraint sets $H_{\text{EUD},t}$ are always convex sets in the dose vector space, since they are level sets (i.e., sets on which the function values are smaller or equal to some fixed real constant) of the convex functions $E_t(h)$ for OAR (with $\alpha_t \geq 1$), or of the convex functions $-E_t(h)$ for targets (with $\alpha_t < 0$).

In the radiation intensity space, the most prominent constraint is the non-negativity of the intensities, described by the set

$$X_+ = \{x \in R^J \mid x_j \geq 0, \text{ for all } j = 1, 2, ..., J\}.$$

Thus, we have a MSFP, where some constraints (the non-negativity of radiation intensities) are defined in the radiation intensity space R^J and other constraints (the upper and lower bounds on dose and the EUD constraints) are defined in the dose space R^I, and the two spaces are related by a linear transformation D (according to (3)). The problem is formulated as follows.

$$\text{Find } x^* \in X_+ \cap \left(\bigcap_{n=1}^{N} X_n \right) \text{ such that } h^* = Dx^* \text{ and } h^* \in \left(\bigcap_{m=1}^{M} H_m \right)$$

5. CONCLUSIONS

We apply a new parallel iterative method for solving the SCFPP in Hilbert spaces, without prior knowledge of the transfer operator norm, to the MSFP, which plays an important role in medical image reconstruction and signal processing.

The new algorithm is of great theoretical significance since it works not only for the SCFPP and the MSFP but also for other split problems. This is also the direction of further development of the research.

Within the framework of this paper, we present a model of the IMRT based on the MSFP as a practical illustration of the problems. The paper can be expanded to specifically calculate a numerical example of a clinical case of a tumor to demonstrate the practical usefulness of our algorithm.

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TÓM TẮT

Bài toán điểm bất động chung tách và ứng dụng trong mô hình xạ trị điều biến cường độ

Trong bài báo này, chúng tôi áp dụng một phương pháp lặp song song mới để giải bài toán điểm bất động chung tách trong không gian Hilbert mà không cần biết thông tin của chuẩn toán tử chuyển cho bài toán chấp nhận tách đa tập, là bài toán có vai trò quan trọng trong việc tái tạo hình ảnh y tế và xử lý tín hiệu. Cuối cùng, chúng tôi xây dựng một mô hình xạ trị điều biến cường độ dựa trên bài toán chấp nhận tách đa tập để minh hoạ cho tính thực tiễn của các bài toán và ý nghĩa quan trọng của thuật toán.

Từ khoá: Bài toán điểm bất động chung tách; Bài toán chấp nhận tách đa tập; Mô hình xạ trị điều biến cường độ.