Building solutions and algorithms for water resource allocation for a river basin

Chu Hoa Lu [1], Dang Tien Trung[2*], Vu Manh Tuan[1]

[1]Institute of Missile, Academy of Military Science and Technology, 17 Hoang Sam, Nghia Do, Hanoi, Vietnam;
[2]Electric Power University, 235 Hoang Quoc Viet, Nghia Do, Hanoi, Vietnam.
*Corresponding author: trungdt@epu.edu.vn

ABSTRACT

In this study, the author proposes a water allocation model based on the linear programming (LP) method combined with the Simplex algorithm to maximize the overall economic benefits while ensuring social constraints and natural limits of the system. The model is built and applied to real data of two river basins. The calculation results demonstrate the effectiveness of the proposed method, with the ability to quickly determine the optimal solution and easily extend to large-scale water resource management systems.

1. INTRODUCTION

For multi-tributary river basins, the problem of optimal water allocation is always recognized as complex and highly challenging. The main reason lies in the instability of the flow, which is continuously influenced by natural factors such as climate fluctuations, infiltration, evaporation, and water quality degradation. In addition, differences in water demand among socio-economic regions further increase system uncertainty, making the design of an appropriate allocation mechanism even more difficult.

In previous studies, many optimization methods have been extensively applied to address this problem, such as nonlinear programming, dynamic programming, and integrated hydrological–economic models [1, 2, 4]. Each method has its own advantages: nonlinear models can accurately represent the physical and economic relationships of the system; dynamic programming handles temporal variations effectively; and hydrological–economic models support comprehensive multi-objective evaluation [7, 9, 10]. However, these approaches also exhibit notable limitations. Nonlinear models require large datasets, long computation times, and are susceptible to local extrema [5]. Dynamic programming suffers from the "curse of dimensionality," making it difficult to apply to multi-regional systems [4]. Meanwhile, integrated simulation models often lack flexibility and are difficult to calibrate across different basins [5], while recent domestic studies also emphasize the need for automated optimization in water distribution systems [8].

To overcome the above limitations, this study proposes a water allocation optimization model based on the LP method combined with the Simplex algorithm. The LP–Simplex approach offers several notable advantages, including high stability, fast computation, the ability to obtain global optimal solutions, and reduced input data requirements compared with traditional nonlinear methods [1, 3]. This framework enables the maximization of overall economic benefits while still ensuring social equity and compliance with the natural constraints of the basin.

2. BUILDING A WATER RESOURCE ALLOCATION ALGORITHM

2.1. Introduction

Consider a river basin with multiple branches that generate water sources and several regions

that require water for domestic use, industrial production, agriculture, electricity generation, and other activities. Allocating water to simultaneously satisfy economic and social requirements is a key challenge. These issues have been approached from various perspectives, and in some cases, allocation decisions are influenced by subjective factors from those in administrative roles. This article proposes a solution and algorithm designed to support more scientific and objective decision-making in the allocation process.

Suppose a river basin has **m** source branches flowing into the main branch (figure 1). Each branch may supply water to a specific water-using area, while the main branch may deliver water to certain consuming parts through irrigation channels or pumping stations.
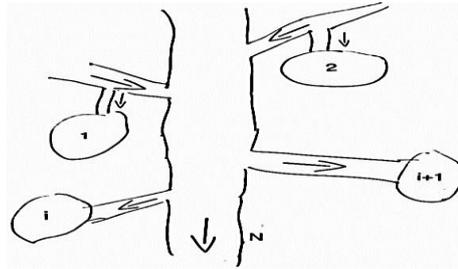

Figure 1. *River basin.*

Receiving water resources and generating economic profits, the profit obtained from each unit of water (km^3) differs across economic units. Let the profit per km^3 of water in economic zone i be a_i, and the amount of water supplied to the socio-economic zone i be x_i. Then, the total profit of the river basin is given by:

$$J = \sum_{i=1}^{N} a_i * x_i \tag{1}$$

From an economic point of view, we always want the total value J to be as large as possible. However, the available water resources x_i are limited, and the economic capacity of each region is also constrained. We now construct the equations that describe these limitations.

The total water resources of a river basin are always less than the basin's total annual water volume, that is:

$$\sum_{i=1}^{N} x_i \leq b_1 \tag{2}$$

The value b_1 can be determined from hydrometeorological statistical data. The values x_i are also constrained by the capacity limits of each tributary's catchment, i.e:

$$x_i \leq c_i \tag{3}$$

The water flows of the tributaries may exhibit infiltration relationships, meaning that the values c_i include not only the natural inflow from the basin's streams but also additional contributions resulting from water infiltration from other areas when the flow is formed:

$$c_i = c_i^* + \mu_{ij} x_j \tag{4}$$

Here, the value c_i^* represents the natural inflow from rivers and streams, while μ_{ij} denotes the osmosis coefficient describing the transfer of water from region j to region i (this occurs when region j uses water but discharges it into branch i instead of its own branch). From an economic perspective, water resources can be concentrated in areas with large coefficients a_i. However, this is not socially feasible, as also emphasized in equity-based water allocation and intergenerational fairness studies [6, 8]. Therefore, the following inequalities must always be satisfied:

$$x_i \geq e_i \tag{5}$$

Here, e_i represents the regulations set by the state to ensure social stability. For example, an economic zone i cultivates a crop with low economic value, yet the local customs and soil conditions are only suitable for that crop. To ensure the livelihood of residents in that area, a minimum amount of water must be allocated. This is a social issue that every country faces and must address. The allocation problem here is to determine the amount of water to be distributed among socio-economic regions so that the total J in expression (1) is maximized, while respecting the limited total supply defined by constraints (2), (3), (4), and (5).

2.2. Building a water resource allocation algorithm

The problem presented in section 2.1 indicates that it falls within the class of LP problems, a framework widely used in operations research and resource allocation applications [2]. According to [3], a LP problem is the task of finding a vector p such that:

$$\underline{p}^* = argmin\left(\underline{a}^T . \underline{p}\right)$$
$$\underline{p} \in P \tag{6}$$

With
$$P = \left\{\underline{p} \in R^n \Big| \underline{C}_i^T \underline{p} \leq d_i \ i = 1,2 \dots m \text{ and } p_j \geq 0, j = 1, 2, \dots, n\right\} \tag{7}$$

The associated vectors and parameters are:
$$\begin{cases} \underline{a} = (a_1, a_2, \dots, a_n)^T \\ \underline{C}_i^T = (c_{i1}, c_{i2}, \dots, c_{in}) \\ \underline{d}_i \in R, i = 1, 2, \dots, m \end{cases} \tag{8}$$

To solve the LP problems (6), (7), and (8), we can, following [1, 3, 5], reduce them to a standard form as follows:

$$\underline{\hat{p}}^* = argmin(\underline{\hat{a}}^T \hat{p})$$
$$\hat{p} \in P \tag{9}$$

Subject to:
$$\widehat{P} = \left\{\hat{p} \in R^{n+m} \Big| \widehat{C}_i^T . \hat{p} = d_i, i = 1,2, \dots, m \text{ and } p_j \geq 0, j = 1,2, \dots, n+m\right\} \tag{10}$$

Where: $\widehat{p} = \left(\underline{p}^T, p_{n+1}, p_{n+2}, \dots, p_{n+m}\right)^T = (p_1, p_2, \dots p_n, p_{n+1}, \dots, p_{n+m})^T$

$$\underline{\hat{a}}^T = \left(\underline{a}^T, 0, \dots, 0\right) \text{ and } \underline{\hat{c}}_i^T = \left(\underline{c}_i^T, 0, \dots, 0, 1, 0, \dots 0\right)$$

We have $\hat{c}_i^T$ with 1st element added at the $(n + i)^{th}$ position. Together with c_i^T, such that the inequality: $\underline{c}_i^T \underline{p} \leq d_i, i = 1, 2, \dots, m$ can be modified following the equation:

$$\underline{c}_i^T \underline{p} + p_{n+i} = d_i \text{ with } (p_{n+i} \geq 0); \ \underline{c}_i^T \widehat{p} = d_i \text{ and } p_{n+i} \geq 0, i = 1, 2, \dots, m$$

For the standard problem, the optimal solution vector $\hat{p}^*$ can be determined explicitly using the Simplex method [1, 3, 5]. In the problem of water resource allocation, there is a constraint (5). This is a lower-bound inequality that contradicts the standard LP form. To overcome this issue, we introduce a new variable as follows:

$$p_i = x_i - e_i \tag{11}$$

It is easy to see that, if $p_i \geq 0$ then satisfies expression (5). From (11) we have:

$$x_i = p_i + e_i \tag{12}$$

Substituting (12) into (1), the following equation is obtained:

$$J = \sum_{i=1}^{N} a_i x_i = \sum_{i=1}^{N} a_i p_i = \sum_{i=1}^{N} a_i e_i \tag{13}$$

Since e_i is a known value, the term $\sum a_i e_i$ is therefore a fixed constant. Thus, maximizing J with respect to x_i, is equivalent to maximizing:

$$J = \sum_{i=1}^{N} a_i p_i \rightarrow \max \tag{14}$$

Inequality (2) then becomes:

$$\sum_{i=1}^{N} x_i = \sum_{i=1}^{N} p_i + \sum_{i=1}^{N} e_i \leq b_1 \tag{15}$$

Using (15), inequality (2) can be rewritten as:

$$\sum_{i=1}^{N} p_i \leq b_1 - \sum_{i=1}^{N} e_i \tag{16}$$

Define:

$$d_1 = b_1 - \sum_{i=1}^{N} e_i \tag{17}$$

So that (16) becomes:

$$\sum_{i=1}^{N} p_i \leq d_1 \tag{18}$$

From inequality (3), we have: $\qquad p_i + e_i \leq c_i \tag{19}$

Which is equivalent to: $\qquad p_i \leq c_i - e_i \tag{20}$

Introduce the notation: $\qquad d_{i+1} = c_i - e_i \tag{21}$

Yielding the bound: $\qquad p_i \leq d_{i+1} \tag{22}$

Substitute x_i from (12) into (4) gives: $c_i = c_i^* + \mu_{ij} \left(p_j + e_j \right) \tag{23}$

Which can be rearranged to: $\qquad \mu_{ij} p_j + \mu_{ij} e_j = c_i - c_i^* \tag{24}$

Thus, the water resource allocation problem can be formulated as a LP problem. Specifically, we seek the values p_i such that:

$$J = \sum_{i=1}^{N} a_i p_i \rightarrow \max \tag{25}$$

Subject to:

$$\sum_{i=1}^{N} p_i \leq d_1 \tag{26}$$

$$p_i \leq d_{i+1} \tag{27}$$

$$\mu_{ij} p_j + \mu_{ij} e_j = c_i - c_i^* \tag{28}$$

The problem expressed by equations (25), (26), (27), (28) corresponds to a standard-form LP model, for which the Simplex method can be applied to obtain the optimal solution. The following section illustrates the proposed solution and algorithm through a specific example. Consider a river basin that includes a single tributary and a main river branch, as shown in figure 2.
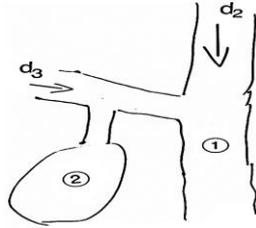
Figure 2. *Two-river basin.*

Suppose region (2) is an ethnic minority area that cultivates a low-profit crop (potato). However, due to ethnic and cultural factors, this region must be provided each year with a guaranteed minimum amount of water d_1 to ensure basic living conditions. Accordingly, the minimum water requirement is:

$$x_2 \geq d_1 \qquad (29)$$

The amount of water flowing into branch 2 in a year is d_3, while the inflow into branch 1 is d_2. Below branch 1 lies an area with a hydroelectric plant. From the diagram, the available water for irrigation is given by:

$$x_1 = d_3 - x_2 + d_2 + \mu x_2 \qquad (30)$$

Here, μ is the osmosis coefficient representing the portion of water transferred from region (2) to region (1). When one unit of water is supplied, the economic benefit in region (1) is a_1, and in region (2) is a_2. The total profit is therefore:

$$J = a_1 x_1 + a_2 x_2 \qquad (31)$$

From (30) we obtain:

$$x_2 = \frac{d_3 - x_1 + d_2}{1 - \mu} \qquad (32)$$

Substituting (32) into (31) gives:

$$J = a_1 x_1 + a_2 \frac{d_3 - x_1 + d_2}{1 - \mu} \qquad (33)$$

This expression can be rewritten as:

$$J = \frac{(d_3 + d_2)a_2}{1 - \mu} + a_1 x_1 - a_2 \frac{x_1}{1 - \mu} \qquad (34)$$

or equivalently:

$$J = \frac{(d_3 + d_2)a_2}{1 - \mu} + \left(a_1 - \frac{a_2}{1 - \mu}\right)x_1 \qquad (35)$$

Let:

$$\alpha = \left(a_1 - \frac{a_2}{1 - \mu}\right) \qquad (36)$$

Maximum on the feasible domain x_1. The feasible domain is usually: $x_1 \in [x_{1,min}, x_{1,max}]$ with $x_{1,min} = d_1$ and $x_{1,max} = \min\{c_1, d_1 + d_3, b_1 - \sum_{i \neq 1} d_i\}$.

3. SIMULATION TEST

In this section, the paper conducts a simulation test of the algorithm presented in section 2 for a basin consisting of two river branches. A data set and the corresponding mathematical model for the water allocation problem are established for this two-branch basin. The model is formulated using the LP method to maximize economic efficiency while simultaneously ensuring the imposed social constraints.

3.1. Input data

Parameter	Value 1	Value 2	Value 3	Value 4	Value 5
Amount of water entering branch 1 (d_2) (km^3/ year)	0,70	0,85	0,90	0,95	1,10
Amount of water entering branch 2 (d_3) (km^3/ year)	0,50	0,65	0,70	0,75	0,90
Osmosis coefficient from zone 2 to zone 1 (α)	0,20	0,20	0,20	0,20	0,20
Marginal benefit of region 1 (a_1) (million/ year /km^3)	120	120	120	120	120
Marginal benefit of region 2 (a_2) (million/ year /km^3)	60	60	60	60	60
Social constraints - minimum level 2 region (d_1) (km^3)	0,30	0,30	0,30	0,30	0,30
Limit the total water volume of the whole basin (km^3)	1,20	1,50	1,60	1,70	2,00

3.2. Decision variables

x_1: Water supply for region 1 (km^3/ year); x_2: Water supply for region 2 (km^3/ year).

3.3. Objective function

Maximize: $J = 120. x_1 + 60. x_2$.

3.4. Constraints

Table 2. *Constraints.*

Constraints	Value 1	Value 2	Value 3	Value 4	Value 5
Branch 2 water limit	$x_2 \leq 0,50$	$x_2 \leq 0,65$	$x_2 \leq 0,70$	$x_2 \leq 0,75$	$x_2 \leq 0,90$
Circulation coefficient from zone 2 to zone 1	$x_1 \leq 0,70$ $+ 0,20. x_2$	$x_1 \leq 0,85$ $+ 0,20. x_2$	$x_1 \leq 0,90 +$ $0,20. x_2$	$x_1 \leq 0,95 +$ $0,20. x_2$	$x_1 \leq 1,10 +$ $0,20. x_2$
Total water limit of the whole basin	$x_1 + x_2$ $\leq 1,20$	$x_1 + x_2$ $\leq 1,50$	$x_1 + x_2$ $\leq 1,60$	$x_1 + x_2$ $\leq 1,70$	$x_1 + x_2$ $\leq 2,00$
Social constraints - minimum water zone 2	$x_2 \geq 0,30$	$x_2 \geq 0,30$	$x_2 \geq 0,30$	$x_2 \geq 0,30$	$x_2 \geq 0,30$
Non-negativity condition	$x_1 \geq 0,$ $x_2 \geq 0$	$x_1 \geq 0,$ $x_2 \geq 0$	$x_1 \geq 0,$ $x_2 \geq 0$	$x_1 \geq 0,$ $x_2 \geq 0$	$x_1 \geq 0,$ $x_2 \geq 0$

3.5. Optimal solution (sample calculation)

Since $a_1 > a_2$, the system naturally prioritizes supplying more water to zone 1. When all constraints are satisfied, the optimal solution is determined as follows:

Table 3. *Optimal solution.*

Optimal solution	Value 1	Value 2	Value 3	Value 4	Value 5
Optimal solution x_1^* (km^3/ year)	0,78	0,96	1,02	1,09	1,25
Optimal solution x_2^* (km^3/ year)	0,42	0,54	0,58	0,61	0,75
Maximum target value J^*(million/ year)	$\approx 119,0$	$\approx 142,8$	$\approx 156,97$	$\approx 171,0$	$\approx 195,0$

3.6. Sensitivity analysis

+ If α decreases to 0 (no water circulation), the amount of water available for zone 1 is reduced, and the value of J decreases significantly.

+ If d_1 increased (e.g., 0.40), more water is allocated to zone 2, which may slightly increase J when α is sufficiently high.

+ If a_2 increased (e.g., to 80), the model naturally tends to allocate more water to zone 2.

As the total water volume and the inflows to the branches increase, the allocation model prioritizes supplying more water to region 1, where the marginal benefits are higher, while still

ensuring the social constraints required for region 2. The allocation ratio x_2^*/x_1^* between the two regions remains stable, fluctuating between 0.54 and 0.60, reflecting the ability of the LP model to balance economic efficiency with social requirements. The objective function J^* increases almost linearly with the total water volume, indicating that the marginal efficiency of water in region 1 remains high. The model consistently satisfies the social constraints for region 2, $x_2 \geq 0,30$, without any violations across all scenarios. With this dataset, the LP approach can be effectively applied to water allocation problems in both irrigation and hydropower. The model can also be extended to additional branches and support the integrated planning of river basins.

3.7. Analysis and evaluation

The results of the water allocation problem indicate that the model operates stably and reliably, accurately capturing the hydro-economic characteristics of the two river basins. The detailed evaluation is presented as follows:

- *Theoretical compatibility:* The optimal solution fully satisfies all constraints of the LP model. The obtained results lie entirely within the feasible domain and achieve the global maximum of the objective function, thereby confirming the correctness and consistency of the mathematical formulation.

- *Practicality:* The allocation results indicate that the amount of water supplied to zone 1 is greater than that of zone 2, which is consistent with the actual economic characteristics when zone 1 has a higher marginal benefit coefficient $a_1 > a_2$. This allocation pattern helps maximize the total economic benefit while still meeting the social requirement of ensuring the minimum water supply for zone 2.

- *Differences from previous studies:* Compared with nonlinear or dynamic programming methods, LP models give equivalent results in efficiency but offer clear advantages in computational speed, simplify the modeling process, and reduce the risk of falling into local extrema.

- *Practical benefits:* The model can be integrated into Decision Support Systems (DSS) or water-distribution management software in irrigation and hydropower. This integration enables managers to quickly determine optimal allocation plans by season or hydrological year, thereby supporting sustainable water-resource planning at both the basin and national scales.

4. CONCLUSIONS

The paper proposed an optimal water allocation model in a multi-branch river basin by adding a new variable (formula 11), thereby reducing the problem to a LP model combined with the Simplex algorithm. The model allows for determining a reasonable distribution of water among usage areas, ensuring maximum economic benefits while still meeting social requirements and complying with the natural limits of the system. The results of calculations with real data show that the model achieves stable optimal solutions, fast processing time, and high accuracy, demonstrating the effectiveness and feasibility of the proposed method in water resources planning and management. In addition, the model can be extended to multi-objective problems or more complex basin systems, contributing to supporting automatic decision-making in water resources management at regional and national scales.

REFERENCES

[1]. G. B. Dantzig, *"Linear Programming and Extensions"*, Princeton University Press, (1963).
[2]. F. S. Hillier and G. J. Lieberman, *"Introduction to Operations Research"*, McGraw-Hill, (2014).
[3]. Nguyễn Doãn Phước, *"Tối ưu hóa trong điều khiển và điều khiển tối ưu"*, Nhà xuất bản Bách khoa Hà Nội, (2016) (in Vietnamese).
[4]. K. Deb, *"Optimization for Engineering Design: Algorithms and Examples"*, PHI Learning, (2012).
[5]. Lingo Systems, *"User's Guide for Linear and Nonlinear Optimization"*, (2021).

[6]. Lv, Chunlan, Yanli Chen, and Qian Huang, *"Intergenerational equity based optimization model joint of water allocation and water quality management"*, Scientific Reports, vol. 15, no. 1, p. 19569, (2025).

[7]. Iban Ortuzar Fernández, et al., *"Balancing Efficiency and Inequality in a Non-Linear MultiRegional Water Allocation Optimization Model"*, (2025).

[8]. Tạ Quốc Huy, *"Tự động hoá tính toán tối ưu cho hệ thống phân phối nước"*, Tạp chí Tự động hoá Ngày nay, (2023) (in Vietnamese).

[9]. Y. Li, B. Xu, X. Zhang, *"Study on the optimization of multi-objective water resources allocation in the Henan Yellow River water supply zone"*, Water, vol. 15, no. 22, p. 4009, (2023).

[10]. B. Tao, L. Zhao, H. Wang, *"Optimizing multi-scenario water resource allocation in reservoirs considering trade-offs between water demand and ecosystem services"*, Water, vol. 16, no. 4, p. 563, (2024).

TÓM TẮT

Xây dựng giải pháp và thuật toán phân bổ nguồn nước cho một lưu vực dòng sông

Trong nghiên cứu này, tác giả đề xuất mô hình phân bổ nước dựa trên phương pháp quy hoạch tuyến tính (LP) kết hợp thuật toán Simplex nhằm tối đa hóa lợi ích kinh tế tổng thể, đồng thời đảm bảo các ràng buộc xã hội và giới hạn tự nhiên của hệ thống. Mô hình được xây dựng và áp dụng cho dữ liệu thực tế của lưu vực hai nhánh sông. Kết quả tính toán chứng minh tính hiệu quả của phương pháp đề xuất, với khả năng xác định nhanh nghiệm tối ưu và dễ dàng mở rộng cho các hệ thống quản lý tài nguyên nước quy mô lớn.

Từ khóa: Quy hoạch tuyến tính; Thuật toán Simplex; Phân bổ nguồn nước; Tối ưu hóa; Lưu vực sông; Quản lý tài nguyên nước.