MODELING OF ELECTROSTATIC PROBLEM VIA A SUBPROBLEM FINITE ELEMENT METHOD

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Abstract: *This work deals with the modeling of electrostatic problem by means of a subproblem finite element method. The aim of this paper is to present naturally the distributions of the electric scalar potential and the electric field in regions existing the various voltages or difference of voltages. The method permits to simultaneously use both node and edge elements in function spaces of the studied problem. Indeed, the distributed fields (e.g. electric scalar potential and electric field) are presented in the same function space and a week formulation written for both nodal and edge finite elements.*

Keywords: Electrostatics; Electric scalar potential; Electric field; Subproblem method (SPM); Weak formulation.

1. INTRODUCTION

In practice, in order to compute and simulate the distribution of electric fields, many authors [1, 2] have applied a finite element method (FEM) to present floating potentials in various electromagnetic problems, where the gradient is considered as a physical field (e.g. $\boldsymbol{e}$ = - grad v, where $\boldsymbol{e}$ is the electric field and v is the electric scalar potential). These are local fields defined in the studied domain with various properties.

Many papers have recently developed a subproblem method (SPM) for modeling of magnetostatic and magneto dynamic problems in the frequency domain [3-5]. The aim of this method is to split a full domain into sub-domains with a series of changes. In the strategy SPM, a first problem with simplified domain (e.g. air gap, thin region…) is solved, and then the next problem with a conducting region or the air is added without considering previous domains anymore. The final solution is presented as a superposition of the subproblem (SP) solutions. Each SP is constrained by interface and boundary conditions (BCs) that express the changes of material characteristics in studied domains.

In this paper, the SPM is extended for calculating the distributions of the electric scalar potential and electric field in electrostatic problems, where a suitable BC on parts of the boundary of the studied domain can be defined as a homogeneous BC for the tangential component of the associated physical vector fields. i.e. $\boldsymbol{n} \times \boldsymbol{e}_i|_{\Gamma_{e,i}} = 0, \quad \boldsymbol{n} \cdot \boldsymbol{d}_i|_{\Gamma_{d,i}} = 0$, for $\boldsymbol{d}_i$ being the electric flux density. The method is highlighted and validated on a test problem.

2. DEFINITION OF THE ELECTROSTATIC PROBLEM

A canonical static problem presented in Tonti diagram [2], to be solved at step i of the problem, is defined in a domain Ω, with boundary $\partial\Omega = \Gamma = \Gamma_e \cup \Gamma_d$. Electrostatics consists of the study of the space distribution of electric field $\boldsymbol{e}$ due to the distribution of electric charges. In this case, $\partial_t \boldsymbol{b}_i = 0$, and thus $\operatorname{curl} \boldsymbol{e}_i = 0$, the set of Maxwell's equations together with the electric constitutive law are [2-6]

$$\operatorname{curl} \boldsymbol{e}_i = 0,\ \operatorname{div} \boldsymbol{d}_i = q_i,\ \boldsymbol{d}_i = \epsilon_i \boldsymbol{e}_i, \tag{1a-b-c}$$

$$\boldsymbol{n} \times \boldsymbol{e}_i|_{\Gamma_{e,i}} = 0, \quad \boldsymbol{n} \cdot \boldsymbol{d}_i|_{\Gamma_{d,i}} = 0, \tag{2a-b}$$

where $\boldsymbol{e}_i$ is the electric field (V/m), $\boldsymbol{d}_i$ is the electric flux density (C/m^2), q_i is the electric charge density (C/ m^2) and ϵ_i is the electric permittivity (F/m). The global constrains related to total charge and potential difference can be also introduced in [1]. In the equation (1a), $\boldsymbol{e}_i$ is defined as the gradient of a scalar potential v_i, i.e.

$$\boldsymbol{e}_i = -\operatorname{grad} v_i. \tag{3}$$

Combining (3) and (1 b), one gets a Poisson equation:

$$-\mathrm{div}(\epsilon_i \mathrm{grad}\, v_i) = q_i. \tag{4}$$

In charge-free regions, the electric scalar potential v_i satisfies Laplace equation:

$$-div(grad\, v_i) = 0. \tag{5}$$

In order to ensure that the solution in (5) is unique, the Dirichlet BC for the potential and Neumann BC for the electric field must be imposed on the surface.

When $\epsilon_i = \epsilon_0$ throughout space, the electric scalar potential v_i in terms of the electric charge density q_i is given by

$$v_i(x) = \frac{1}{4\pi\epsilon_0}\int_{R^3}\frac{q_i(y)}{|x-y|}dy. \tag{6}$$

The electrostatic problem fits naturally in the Tonti diagram (fig. 1).

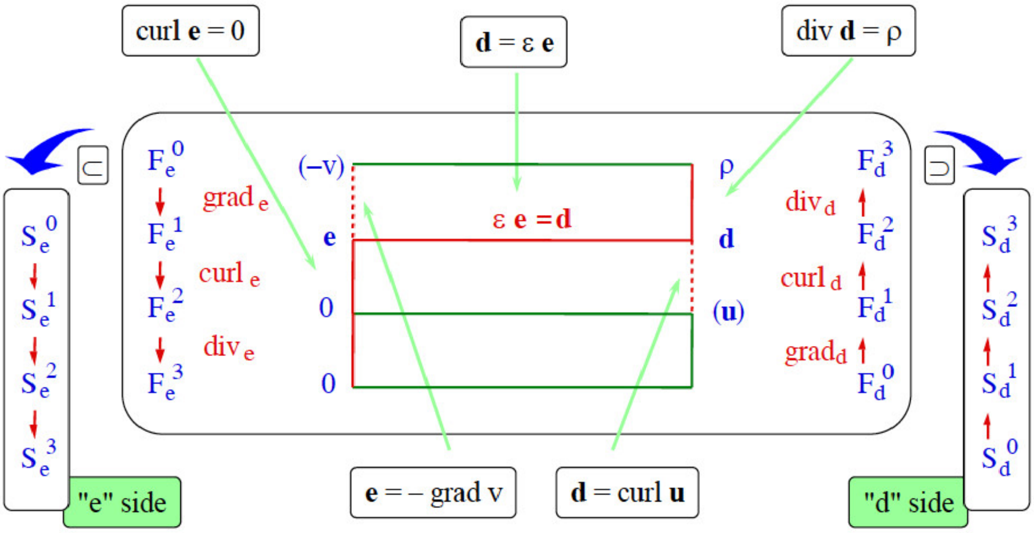

***Figure 1**. Tonti diagram [2].*

The unknown fields $\boldsymbol{d}_i \in F^0_{h,i}(div;\, \Omega_i)$ and $\boldsymbol{e}_i \in \boldsymbol{F}_e\,(curl;\, \Omega_i)$ solutions of (1a-b) together with the constitutive law (1c), and the electric scalar potential $v_i \in \boldsymbol{F}^1_{\boldsymbol{e}}(\Omega_i)$, which verifies (3), can be introduced in the Tonti diagram. The Tonti diagram for the electrostatic problem is given

$$\begin{tikzcd} 0 & \boldsymbol{e}_i \arrow[l, "\mathbf{curl}_e"'] \arrow[d, leftrightarrow, "\epsilon"] & v_i \arrow[l, "\mathbf{grad}_e"'] \\ & \boldsymbol{d}_i \arrow[r, "\mathrm{div}_h"] & q_i. \end{tikzcd} \tag{7}$$

3. FINITE ELEMENT WEAK FORMULATION

3.1. Green formulas theory

The following notations are used for integration of products of scalar or vector fields over a volume Ω_i or on a surface Γ_i, where $\mathcal{L}^2$ and $\boldsymbol{\mathcal{L}}^2$ are the spaces of square-summable scalar and vector functions:

$$(u_i, v_i) = \int_{\Omega_i} u_i\,(\mathrm{x}) \cdot v_i(\mathrm{x})d\mathrm{x},\ \forall\, u_i, v_i \in \boldsymbol{\mathcal{L}}^2(\Omega_i), \tag{8}$$

$$(\boldsymbol{u}_i, \boldsymbol{v}_i) = \int_{\Omega_i} \boldsymbol{u}_i\,(\mathrm{x}) \cdot \boldsymbol{v}_i(\mathrm{x}) dx,\ \forall\, \boldsymbol{u}_i, \boldsymbol{v}_i \in \mathcal{L}^2(\Omega_i), \tag{9}$$

$$\langle u_i, v_i\rangle_{\Gamma_i} = \int_{\Gamma_i} u_i\,(\mathrm{x}) \cdot v_i(\mathrm{x}) d\boldsymbol{s},\quad \langle \boldsymbol{u}_i, \boldsymbol{v}_i\rangle_{\Gamma_i} = \int_{\Gamma_i} \boldsymbol{u}_i\,(\mathrm{x}) \cdot \boldsymbol{v}_i(\mathrm{x}) d\boldsymbol{s}. \tag{10a-b}$$

A first relation of vectorial analysis

$$\boldsymbol{u}_i.\,\mathrm{grad}\, v_i + v_i\, \mathrm{div}\, \boldsymbol{u}_i = \mathrm{div}\, (\boldsymbol{u}_i, \boldsymbol{v}_i), \tag{11}$$

integrated in the studied domain Ω_i, after applying the divergence theorem, gives the **Green formula** said of kind **grad-div** in Ω_i, i.e.

$$(\boldsymbol{u}_i, \mathrm{grad}\, v_i)_{\Omega_i} + (\mathrm{div}\, \boldsymbol{u}_i, v_i)_{\Omega_i} = \langle v_i, \boldsymbol{n}.\boldsymbol{u}_i\rangle_{\Gamma_i},$$
$$\forall\, \boldsymbol{u}_i \in \boldsymbol{H}^1(\Omega),\ \forall\, v_i \in H^1(\Omega), \tag{12}$$

where $H^1(\Omega)$ and $\boldsymbol{H}^1(\Omega)$ are function spaces built for the scalar and vector fields, respectively. Notations $(\cdot, \cdot)$ and $< \cdot, \cdot >$ are respectively a volume integral in and a surface integral of the product of scalar or vector fields [5].

Another relation of vectorial analysis

$$\boldsymbol{u}_i.\,\mathrm{curl}\, \boldsymbol{v}_i - \mathrm{curl}\, \boldsymbol{u}_i.\boldsymbol{v}_i = \mathrm{div}\, (\boldsymbol{v}_i \times \boldsymbol{u}_i), \tag{13}$$

integrated in the studied domain Ω_i, after applying the divergence theorem, gives the Greenformula said of kind **curl-curl** in Ω_i, i.e.

$$(\boldsymbol{u}_i, \mathrm{curl}\, \boldsymbol{v}_i)_{\Omega_i} - (\mathrm{curl}\, \boldsymbol{u}_i, \boldsymbol{v}_i)_{\Omega_i} = \langle\, \boldsymbol{n} \times \boldsymbol{u}_i, v_i\rangle_{\Gamma_i},$$
$$\forall\, \boldsymbol{u}_i, \boldsymbol{v}_i \in \boldsymbol{H}^1(\Omega). \tag{14}$$

Note that the surface integral term appearing in this last formula can take the following similar forms:

$$\langle\, \boldsymbol{n} \times \boldsymbol{u}_i, \boldsymbol{v}_i\rangle_{\Gamma_i} = \langle\, \boldsymbol{u}_i \times \boldsymbol{v}_i, \boldsymbol{n}\rangle_{\Gamma_i} = -\langle\, \boldsymbol{v}_i \times \boldsymbol{n}, \boldsymbol{u}_i\rangle_{\Gamma_i}. \tag{15}$$

It is possible to define a generalized Green formula [1]

$$(\boldsymbol{\mathcal{L}}\boldsymbol{e}_i, v_i\,)_{\Omega_i} - (\boldsymbol{e}_i, \boldsymbol{\mathcal{L}}^* v_i)_{\Omega_i} = \int_{\Gamma_i} Q\,(\boldsymbol{e}_i, v_i) ds,\ \forall\, \boldsymbol{e}_i \in \mathcal{D}(\boldsymbol{\mathcal{L}}) \text{ and } \forall\, v_i \in \mathcal{D}(\boldsymbol{\mathcal{L}}^*), \tag{16}$$

where $\boldsymbol{\mathcal{L}}$ and $\boldsymbol{\mathcal{L}}^*$ are differential operators of order *n* which act respectively on functions $\boldsymbol{e}_i$ and v_i defined in $\overline{\Omega_i} = \Omega_i \cup \Gamma_i$. Q is a bi-linear function of $\boldsymbol{e}_i$ and v_i. $\boldsymbol{\mathcal{L}}^*$ is called the dual operator of $\boldsymbol{\mathcal{L}}$.

3.2. Canonical electrostatic problem in a weak formulation

Let us consider the Green formulas theory developed in Section 3.1. The electric scalar potential weak formulation obtained from the particularized Maxwell's equation (1a-b) with the constitutive law (1c) was given in (4) [3-6], i.e.

$$(-\epsilon_i \mathrm{grad}\, v_i, \mathrm{grad}\, v_i')_{\Omega_i} - \langle \boldsymbol{n} \cdot \boldsymbol{d}_i, v_i'\rangle_{\Gamma_i} = 0, \forall\, v_i' \in F^1(\Omega_i), \tag{17}$$

where $F^1(\Omega_i)$ is the function space defined on Ω_i containing the basic functions for v_i as well as for the test function v_i' (at the discrete level, this space is defined by nodal FEs.

The surface integral term on Γ_i $\langle \boldsymbol{n} \cdot \boldsymbol{d}_i, v_i'\rangle_{\Gamma_i}$ in (18) can be associated with a global quantity or used for fixing a natural BC (usually homogeneous Neumann BC for a tangent electric field constraint) on a portion $\Gamma_{d,i}$ of the boundary of Γ_i already given in (2 b).

At the discrete level, the filed is generally discretized in a nodal finite element space, defined on a mesh of Ω_i and denoted $S^0(\Omega_i)$ (fig. 1), i.e.

$$v_i = \textstyle\sum_{n \in N} v_{i,n} s_n\,,\ v_i \in S^0(\Omega_i), \tag{18}$$

where N is the set of nodes of Ω_i, s_n is the nodal basis function associated with node n and $v_{i,n}$ is the value of v_i at node n.

4. APPLICATION EXAMPLE

The test problem is a hydrocarbon storage tank given in figure 2, where the distance between two long parallel plates is 10 m (proposed by many authors in [7]). Between the two parallel plates, the density of electric charges and dielectric permittivity are constant. The boundary is fixed at zero potential.

The first test is considered with the electric potential of the below and above plates being $0V$ and $100V$, respectively. The distributions of the electric field and electric scalar potential in a parallel plate due to the voltage of *0V* and the voltage of *100V* on the plates is shown in figure 3 and figure 4. The electric equipotential lines and electric field calculated by the SPM are presented in figure 5 and figure 6. They are checked to be close to the analytic solutions found from the analytic method. The mean errors on the electric scalar potential are lower than 5% for both cases. It can be shown that a very good agreement between two computation methods is observed. This can be also demonstrated that the proposed SPM has been validated.

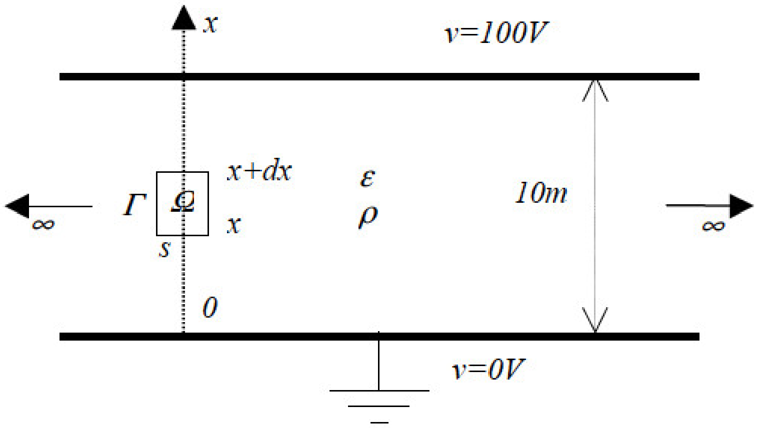

Figure 2. *Geometry of electric charges between two plates [7].*

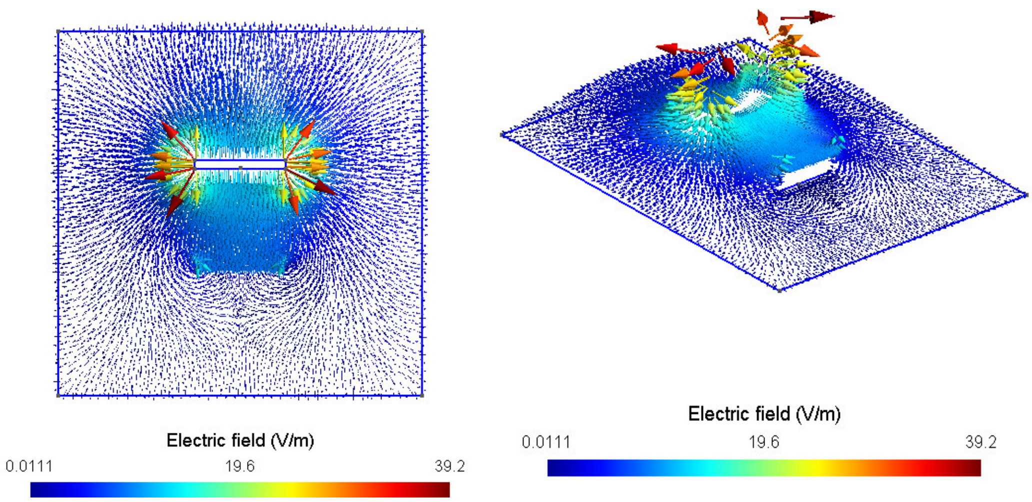

Figure 3. *The distribution of the electric field in a parallel plate due to a voltage difference (0V and 100V) on the plates.*

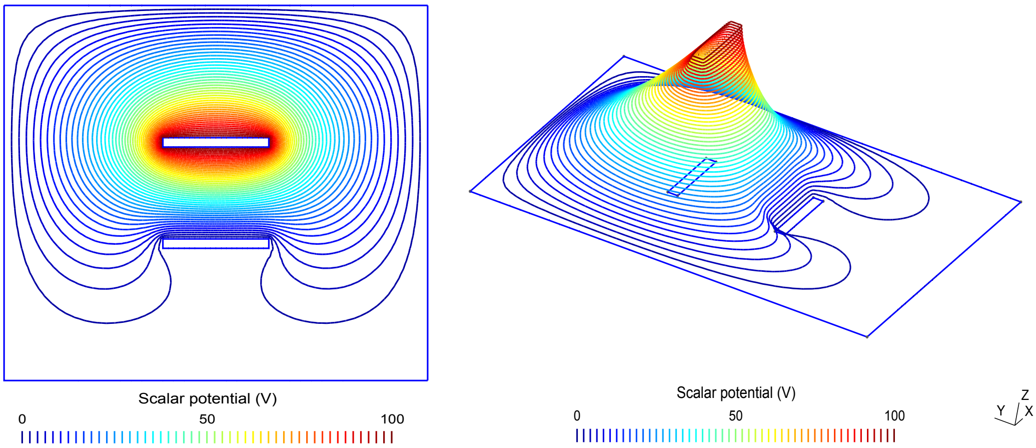

Figure 4. *The distribution of the electric scalar potential in a parallel plate due to a voltage difference (0V and 100V) on the plates.*

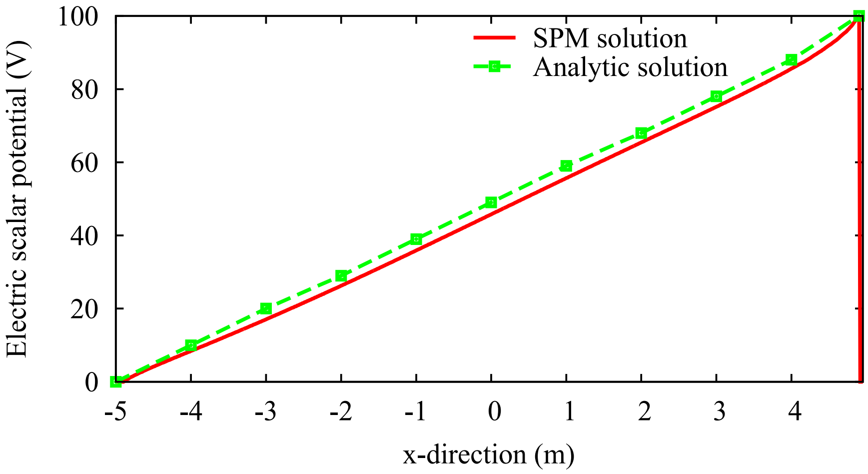

Figure 5. *The distribution of the electric scalar potential between two plates due to a voltage difference (0V and 100V) on the plates.*

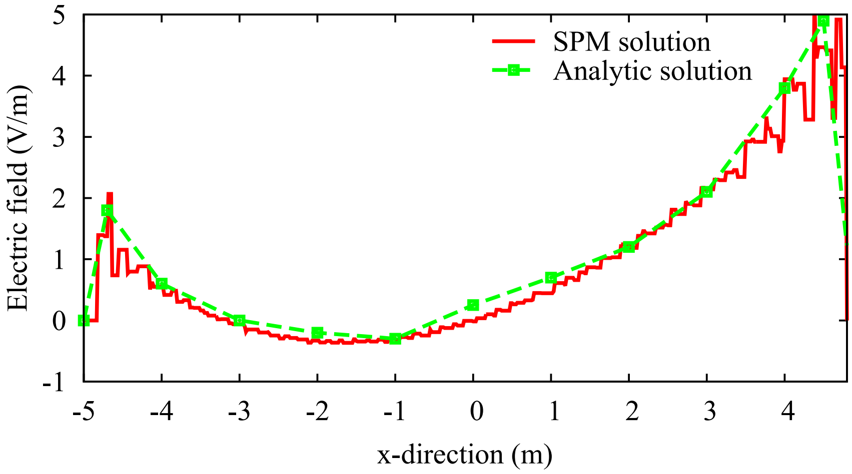

Figure 6. *The distribution of the electric field between two plates due to a voltage difference (0V and 100V) on the plates.*

The second test is solved with the electric potential of the below and above plates being -100 V and 100 V, respectively. The distributions of the electric field and electric scalar potential in a parallel plate due to a voltage difference on the plates are pointed out in figure 7 and figure 8, respectively. The electric field focuses towards the above plate, where the scalar potentials are equally distributed on both sides of two plates.

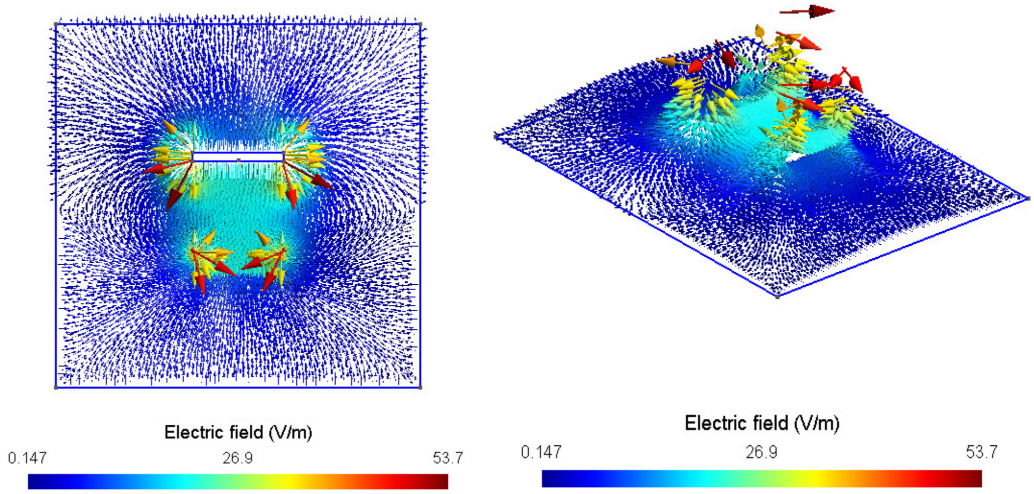

Figure 7. *The distribution of the electric field in a parallel plate due to a voltage difference (-100V and 100V) on the plates.*

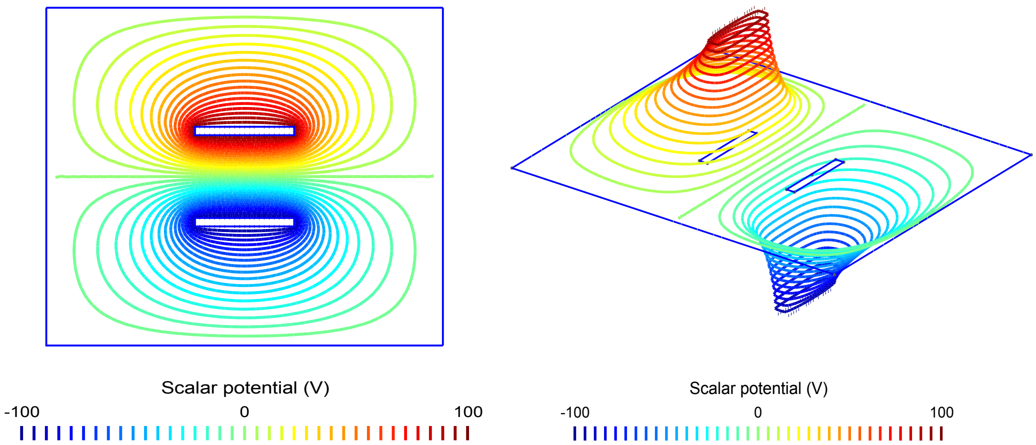

Figure 8. *The distribution of the electric scalar potential in a parallel plate due to a voltage difference (-100V and 100V) on the plates.*

5. CONCLUSION

The presented SPM has been successfully applied to the actual problem for computing the distribution of fields (electric field and scalar potential) with a voltage difference. It has been also applied for the various physical electrostatic problems in the weak formulation. The obtained results of the method are also compared with the solutions defined from the analytic method. The detailed study of the electric field and scalar potential is discretized on the finite element nodes.

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TÓM TẮT

MÔ HÌNH BÀI TOÁN ĐIỆN TỪ TĨNH BẰNG PHƯƠNG PHÁP MIỀN NHỎ HỮU HẠN

Nghiên cứu này phân tích mô hình bài toán điện từ tĩnh bằng phương pháp miền nhỏ hữu hạn. Mục đích của bài báo là trình bày sự phân bố của điện trường và điện thế vô hướng trong vùng mà tồn tại các điện thế thay đổi hoặc điện thế khác nhau. Phương pháp cho phép sử dụng đồng thời cả phần tử cạnh và phần tử nút trong không gian hàm của bài toán nghiên cứu. Thực vậy, sự phân bố của các trường như điện trường và điện thế vô hướng được biểu diễn trong cùng không gian hàm và phương trình yếu nhận được viết cho cả phần tử cạnh và phần tử nút.

Từ khoá: Điện tĩnh; Điện thế vô hướng; Điện trường; Phương pháp miền nhỏ hữu hạn; Phương trình yếu nhận.

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