BUCKLING ANALYSIS OF FG MICRO-BEAM RESTING ON ELASTIC FOUNDATION VIA NONLOCAL STRAIN GRADIENT THEORY

Dang Van Hieu*, Le Quang Duy

Abstract: *In this work, the Euler-Bernoulli beam and the nonlocal strain gradient theories are developed to analyze the buckling behavior of a functionally graded micro-beam resting on an elastic foundation. The governing equation of motion for the functionally graded micro-beam is derived by employing the Hamilton's principle. The analytical expression of critical buckling force for the functionally graded micro-beam with simply supported boundary conditions is obtained in the closed form. The accuracy of the obtained result is verified by comparing the obtained solutions with the published ones. Effects of the power-law index, the material length scale parameter, the nonlocal parameter ,and the coefficients of the elastic foundation on the buckling response of the functionally graded micro-beam are investigated and discussed.*

Keywords: Functionally graded materials; Buckling analysis; Nonlocal strain gradient; Micro-beam.

1. INTRODUCTION

First discovered in 1984 by scientists in Japan, the functionally graded materials (FGMs) have some advantages over the traditional composite materials. FGMs are increasingly being used in the construction of structures that work in environments requiring high-temperature resistance, corrosion resistance, improved stress spreading and inferior stress intensity factors [1-2]. A large number of works analyzing mechanical behaviors of functionally graded (FG) structures have been reported such as beams [3-5], plates [6-8].

With the development of science and technology, micro-/nano-sized structures are applied in many fields, especially in micro-/nano-electromechanical systems (MEMs/NEMs) such as biosensors, micro-sensors, and micro-actuators [9]. Unlike macro-structures, the size-dependent influences on the mechanical behaviors of the micro-/nano-scaled structures are very important and cannot be ignored. Some higher-order elasticity theories that can capture the size-dependent effects on the behaviors of the micro-/nano-scaled structures have been developed. For example, the nonlocal elasticity theory (NET) introduced by Eringen [10, 11] considers the stress at a point as a function of strains at all points in the continuum body, only the stiffness-softening effect is observed by using this theory. By modeling small-scaled materials as atoms with higher-order deformation mechanism instead of collections of points, the strain gradient theory (SGT) [12-16] states that the total stress should consider some additional strain gradient terms. Some versions of the SGT such as the modified couple stress theory (MCST) [17] and the modified strain gradient theory (MSGT) [18] were proposed. Contrary to the NET; the SGT, MCST, and MSGT play the stiffness-hardening effect on responses of the micro-scaled structures. Recently, in 2015, Lim et al. [19] combined the NET and the SGT into a generalized higher-order elasticity theory called the nonlocal strain gradient theory (NSGT). Effects of the two length scales including the material length scale parameter and the nonlocal parameter on the mechanical and physical behaviors of size-dependent structures are described in the NSGT. Many works related to the bending, buckling and vibration behaviors of micro-structures have been published based on these elasticity theories. Li and Hu [20] investigated the buckling behavior of size-dependent nonlinear beams based on the NSGT. The size-dependent nonlinear Euler-Bernoulli and Timoshenko beam models were developed by Li and Hu [21] to study bending and free vibration characteristics of FG beams using the NSGT. A size-dependent sinusoidal shear deformation beam model was developed by Lu et al. [22] employing the NSGT to analyse the free vibration response of nano-beams. The nonlinear free vibration behavior of an FG nano-beam was studied

by Simsek [23] using the Euler-Bernoulli beam theory and the NSGT. The bending, vibration and buckling behaviors of bi-directional FG porous micro-beams with a variable material length scale parameter were investigated by Karamanli and Vo [25] using the MSGT. And recently, using the unified shear deformable beam theory and the MCST, Liu and Zhang [26] studied the nonlinear dynamics response of atwo-directional FG micro-beams with geometrical imperfection.

In this work, the authors focus on analyzing of the buckling response of the FG micro-beam resting on the elastic foundation based on the NSGT. The governing equation of motion for the FG micro-beam is derived by using the Euler-Bernoulli beam theory with the von-Kármán's nonlinear strain-gradient relationship and the Hamilton's principle. The analytical expression of the critical buckling force of the FG micro-beam with simply supported boundary conditions is obtained in the analytical form. Comparing the current result with the published ones shows the accuracy of the obtained solution. Effects of the power-law index, the material length scale parameter, the nonlocal parameter, and the coefficients of the elastic foundation on the buckling behavior of the FG micro-beam are investigated and discussed in detail.

2. MODEL AND FORMULATIONS

An FG micro-beam of length L, width b, and height h is considered in figure 1. The FG micro-beam rests on an elastic foundation with coefficients k_L and k_S corresponding to the Winkler elastic layer and the Pasternak elastic layer, respectively. The FG micro-beam composed of ceramic and metal is considered. The material properties of the FG micro-beam are assumed as functions of the thickness coordinate (z). Based on the rule of mixture, the effective properties of the FG micro-beam including the Young's modulus $E(z)$ and the mass density $\rho(z)$ are expressed as [21, 23]:

$$E(\mathrm{z}) = (E_c - E_m)\left(\frac{z}{h} + \frac{1}{2}\right)^k + E_m \tag{1}$$

$$\rho(\mathrm{z}) = (\rho_c - \rho_m)\left(\frac{z}{h} + \frac{1}{2}\right)^k + \rho_m \tag{2}$$

where E_c and E_m indicate respectively the Young's moduli of ceramic and metal; ρ_c and ρ_m are the mass densities of ceramic and metal, respectively; k is the power-law index which governs the variation of materials in the FG micro-beam.

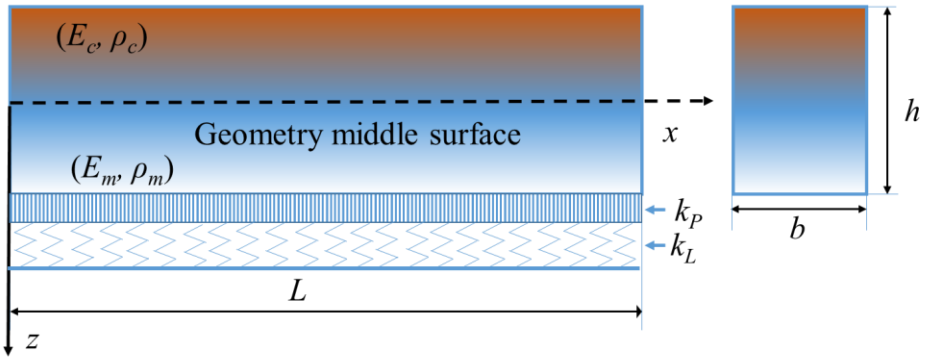


Figure 1. *Modelling of an FG micro-beam resting on an elastic foundation.*

Based on the NSGT [19] and the Euler-Bernoulli beam model with the von-Kármán's assumption about strain-displacement nonlinear relationship, the equilibrium equation of the FG micro-beam resting on an elastic foundation can be extended from the equation obtained by

Simsek [23] as:

$$D_{xx}\left(1-l_m^2\frac{\partial^2}{\partial x^2}\right)\frac{\partial^4 w}{\partial x^4}+k_L\left[w-(ea)^2\frac{\partial^2 w}{\partial x^2}\right]-k_S\frac{\partial^2}{\partial x^2}\left[w-(ea)^2\frac{\partial^2 w}{\partial x^2}\right]+$$
$$\left\{P_0-\frac{A_{xx}}{2L}\int_0^L\left(\frac{\partial w}{\partial x}\right)^2 dx+\frac{A_{xx}}{L}l_m^2\int_0^L\left[\frac{\partial w}{\partial x}\frac{\partial^3 w}{\partial x^3}+\left(\frac{\partial^2 w}{\partial x^2}\right)^2\right]dx\right\}\left[\frac{\partial^2 w}{\partial x^2}-(ea)^2\frac{\partial^4 w}{\partial x^4}\right]=0. \quad (3)$$

where w is the transverse displacement of any point on the geometry middle surface of the FG micro-beam; P_0 is the compressive axial force; ea is the nonlocal parameter describing the influence of the nonlocal stress; l_m is the material length scale parameter introduced to determine the significance of higher-order strain gradient field; A_{xx} and D_{xx} are respectively extensional and bending coefficients which are defined as:

$$A_{xx}=\int_A E(z)dA=b\int_{-h/2}^{h/2}\left[(E_c-E_m)\left(\frac{z}{h}+\frac{1}{2}\right)^k+E_m\right]dz \quad (4)$$

$$D_{xx}=\int_A E(z)z^2dA=b\int_{-h/2}^{h/2}\left[(E_c-E_m)\left(\frac{z}{h}+\frac{1}{2}\right)^k+E_m\right]z^2dz \quad (5)$$

3. BUCKLING ANALYSIS

For buckling analysis, the following dimensionless variables are introduced:

$$\bar{x}=\frac{x}{L},\bar{w}=\frac{w}{L},\ \alpha=\frac{ea}{L},\beta=\frac{l_m}{L},\gamma=\frac{L}{h},P=\frac{P_0L^2}{E_Lbh^3},K_L=\frac{k_LL^4}{E_Lbh^3},\ K_S=\frac{k_SL^2}{E_Lbh^3},\ \bar{z}=\frac{z}{h}. \quad (6)$$

Using (6), the equilibrium equation (3) can be rewritten in the following dimensionless form:

$$\bar{D}_{xx}\left(1-\beta^2\frac{\partial^2}{\partial\bar{x}^2}\right)\frac{\partial^4\bar{w}}{\partial\bar{x}^4}+K_L\left[\bar{w}-\alpha^2\frac{\partial^2\bar{w}}{\partial\bar{x}^2}\right]-K_S\left[\frac{\partial^2\bar{w}}{\partial\bar{x}^2}-\alpha^2\frac{\partial^4\bar{w}}{\partial\bar{x}^4}\right]+$$
$$\left\{P-\frac{\bar{A}_{xx}}{2}\gamma^2\int_0^1\left(\frac{\partial\bar{w}}{\partial\bar{x}}\right)^2 d\bar{x}+\bar{A}_{xx}\beta^2\gamma^2\int_0^1\left[\frac{\partial\bar{w}}{\partial\bar{x}}\frac{\partial^3\bar{w}}{\partial\bar{x}^3}+\left(\frac{\partial^2\bar{w}}{\partial\bar{x}^2}\right)^2\right]d\bar{x}\right\}\left[\frac{\partial^2\bar{w}}{\partial\bar{x}^2}-\alpha^2\frac{\partial^4\bar{w}}{\partial\bar{x}^4}\right]=0. \quad (7)$$

where:

$$\bar{A}_{xx}=\int_{-1/2}^{1/2}\left[\left(\frac{E_c}{E_m}-1\right)\left(\bar{z}+\frac{1}{2}\right)^k+1\right]d\bar{z} \quad (8)$$

$$\bar{D}_{xx}=\int_{-1/2}^{1/2}\left[\left(\frac{E_c}{E_m}-1\right)\left(\bar{z}+\frac{1}{2}\right)^k+1\right]\bar{z}^2d\bar{z} \quad (9)$$

For the simply supported FG micro-beam, the solution of Eq. (7) can be assumed as follows:

$$\bar{w}(\bar{x})=\sum_{n=1}^{\infty}\bar{W}_n\sin(n\pi\bar{x}) \quad (10)$$

where n is the number of half-waves. Substituting the solution (10) into Eq. (7) leads to:

$$\bar{W}_n^2=-\frac{4\bar{D}_{xx}c_2}{\bar{A}_{xx}\gamma^2c_1}+\frac{4P}{\bar{A}_{xx}\gamma^2\lambda^2}-\frac{4K_L}{\bar{A}_{xx}\gamma^2\lambda^4}-\frac{4K_S}{\bar{A}_{xx}\gamma^2\lambda^2} \quad (11)$$

By letting $\bar{W}_n = 0$ in Eq. (11), the dimensionless critical buckling force can be obtained as:

$$P_{cr} = \frac{\bar{D}_{xx}\lambda^2 c_2}{c_1} + \frac{K_L}{\lambda^2} + K_S \tag{12}$$

Equation (12) is the dimensionless critical buckling force for the FG micro-beam resting on the elastic foundation with the simply supported boundary condition based on the NSGT. Considering Eqs. (6) and (9), the physical form of the critical buckling force can be achieved as:

$$\left(P_0\right)_{cr} = \frac{D_{xx}\lambda^2 c_2}{L^2 c_1} + \frac{k_L}{\lambda^2} + k_S = \frac{D_{xx}(n\pi)^2(1+\beta^2 n^2\pi^2)}{L^2(1+\alpha^2 n^2\pi^2)} + \frac{k_L}{(n\pi)^2} + k_S \tag{13}$$

For the homogeneous micro-beam and without the elastic foundation, the critical buckling force (13) becomes:

$$\left(P_0\right)_{cr} = \frac{EI(n\pi)^2(1+\beta^2 n^2\pi^2)}{L^2(1+\alpha^2 n^2\pi^2)} \tag{14}$$

which is the same as the result achieved by Li and Hu [20].

For the classical micro-beam, the critical buckling force (14) reduces to:

$$\left(P_0\right)_{cr} = \frac{EI(n\pi)^2}{L^2} \tag{15}$$

It can be seen that the coefficients of the elastic foundation (k_L and k_s) lead to an increase of the critical buckling force, the critical buckling force increases by increasing the material length scale parameter ($\beta = l_m / L$) or by reducing the nonlocal parameter ($\alpha = ea / L$).

4. NUMERICAL RESULTS AND DISCUSSION

The obtained results are verified by comparing the present results with those of Reddy [24] for the homogeneous beam using the nonlocal elasticicy theory. Note that the result for the nonlocal beam can be recorved from the current result by letting $\beta = 0$. It can be observed that the obtained results are the same as those of Reddy [24] (see table 1). For the nonlocal strain gradient homogeneous micro-beam and without the elastic foundation, the result obtained in this paper is the same as that of Li and Hu [20] (see Eq. (14)).

Table 1. *Comparison of dimensionless critical buckling loads ($\bar{P}_{cr} = P_{cr} \times (L^2/EI)$) of simply supported beams (L/h=20).*

	ea^2						
	0.0	0.5	1.0	1.5	2.0	2.5	3.0
Reddy [24]	9.8696	9.4055	8.9830	8.5969	8.2426	7.9163	7.6149
Present	9.8696	9.4055	8.9830	8.5969	8.2426	7.9163	7.6149

In the below section, numerical results are carried out to study the effect of important parameters on the buckling behavior of the FG micro-beam. A micro-beam composed of Al(metal) and Al_2O_3 (ceramic) is considered. The critical buckling force ratio (the ratio of the critical buckling force to the classical critical buckling force) and the scale ratio (the ratio of the nonlocal parameter to the material length scale parameter) are defined as follows:

$$R_{cr} = \frac{P_{cr}}{(P_{cr})_{classical}}, \chi = \frac{\alpha}{\beta} = \frac{ea}{l_m} \tag{16}$$

where $(P_{cr})_{classical}$ is the classical dimensionless critical buckling force of the metal micro-beam without the elastic foundation.

The effects of the dimensionless material length scale parameter β and the dimensionless nonlocal parameter α on the critical buckling force ratio R_{cr} of the FG micro-beam are presented in figure 2. The result shows that the classical results ($\chi = 0$ and $\alpha = 0$) are equal to the nonlocal strain gradient results if $\chi = 1$. When $\chi < 1$ (i.e., the nonlocal parameter is smaller than the material length scale parameter), the critical buckling force ratio increases as the dimensionless material length scale parameter is increased. On the other hand, the critical buckling force ratio decreases as the dimensionless material length scale parameter is increased if $\chi > 1$ (i.e., the nonlocal parameter is larger than the material length scale parameter). This result can be explained because the FG micro-beam exerts the hardening effect if $\chi < 1$ and the softening effect if $\chi > 1$.

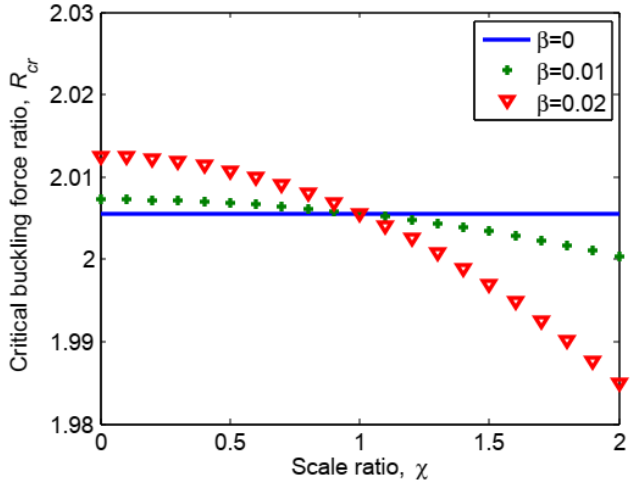


Figure 2. *Variation of the critical buckling force ratio to the scale ratio for some values of the dimensionless material length scale parameter ($k = 5$, $\gamma = 20$, $K_L = 2$, $K_S = 0$ and $n = 1$).*

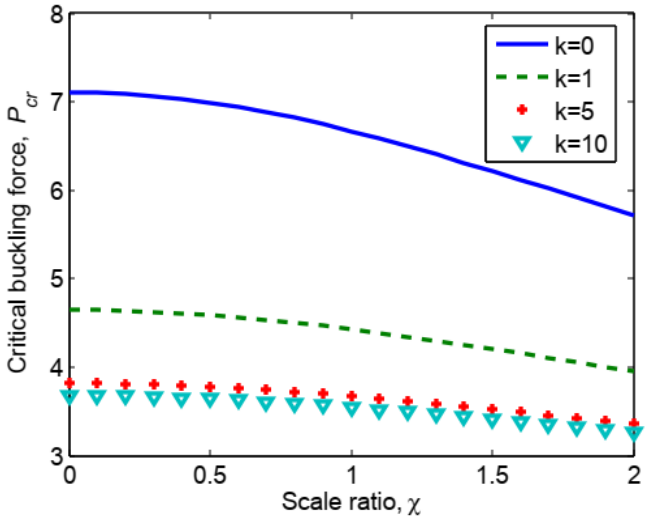


Figure 3. *Variation of the dimensionless critical buckling force to the scale ratio for some values of the power-law index ($\gamma = 20$, $\beta = 0.1$, $K_L = 2$, $K_S = 2$ and $n = 1$).*

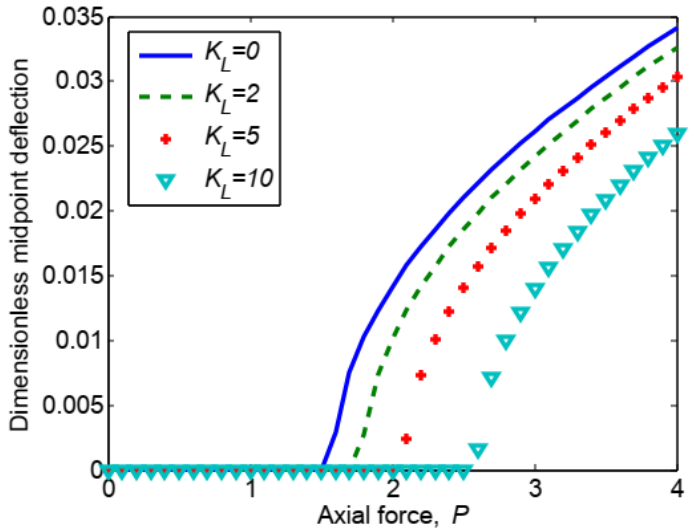


Figure 4. *Buckling deflection at the midpoint of the FG micro-beam for some different values of the dimensionless Winkler parameter ($\gamma = 20$, $\beta = 0.01$, $\chi = 0.5$, $K_S = 0$, $n = 1$ and $k = 3$).*

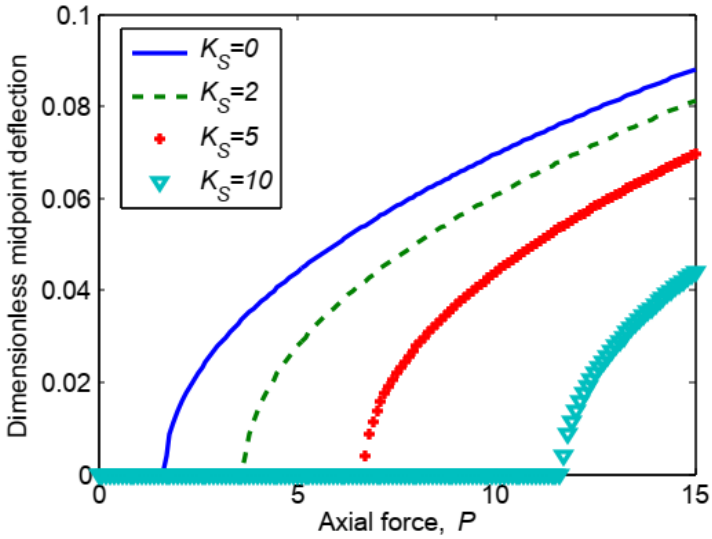


Figure 5. *Buckling deflection at the midpoint of the FG micro-beam for some different values of the dimensionless Pasternak parameter ($\gamma = 20$, $\beta = 0.01$, $\chi = 0.5$, $K_L = 2$, $n = 1$ and $k = 5$).*

Figure 3 depicts the effect of the power-law index k on the buckling behavior of the FG micro-beam. It can be observed that the power-law index k decreases the critical buckling force of the FG micro-beam. The FG micro-beam will be more flexible as the power-law index increases, this leads to a decrease of the critical buckling force. When the power-law index is equal to zero ($k = 0$), it means that the micro-beam is composed entirely of ceramic which makes the micro-beam stiffer; and consequently, the critical buckling force increases.

Effects of the coefficients of the elastic foundation on the critical buckling force of the FG micro-beam can be seen in figures 4 and 5. The dimensionless buckling deflections of the midpoint of the FG micro-beam for some different values of the dimensionless Winkler parameter K_L and the dimensionless Paternak parameter K_S are presented in figures 4 and 5, respectively. It can be concluded that the coefficients of the elastic foundation lead to an increase of the critical buckling force of the FG micro-beam.

5. CONCLUSIONS

Based on the NSGT and the Euler-Bernoulli beam model, the buckling behavior of the FG micro-beam resting on the elastic foundation is investigated in this paper. The FG micro-beam composed of ceramic and metal with the material properties varying in the thickness direction according to the rule-mix distribution is considered. The analytical expression of critical buckling force of the FG micro-beam with simply supported boundary conditions is obtained. Effects of the power-law index, the material length scale parameter, the nonlocal parameter and the coefficients of the elastic foundation on the buckling behavior of the FG micro-beam are investigated and discussed. The important conclusions can be summarized as follows:

- The critical buckling force increases by decreasing the power-law index or by increasing the coefficients of the elastic foundation.

- The material length scale parameter leads to an increase of the critical buckling force, while the nonlocal parameter leads to a decrease of the critical buckling force. Depending on the relationship between the material length scale parameter and the nonlocal parameter, the hardening effect or the softening effect may be arised.

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TÓM TẮT

PHÂN TÍCH ỔN ĐỊNH CỦA DẦM MICRO LÀM BẰNG VẬT LIỆU CƠ TÍNH BIẾN THIÊN TỰA TRÊN NỀN ĐÀN HỒI SỬ DỤNG LÝ THUYẾT ĐỘ DỐC BIẾN DẠNG KHÔNG ĐỊA PHƯƠNG

Trong công trình này, lý thuyết dầm Euler-Bernoulli và lý thuyết độ dốc biến dạng không địa phương được phát triển để phân tích ổn định nén của dầm micro làm bằng vật liệu cơ tính biến thiên tựa trên nền đàn hồi. Phương trình chuyển động của dầm micro được xây dựng bằng cách sử dụng nguyên lý Hamilton. Biểu thức giải tích của lực nén tới hạn đối với dầm micro với hai đầu tựa bản lề được tìm dưới dạng đóng. Độ chính xác của kết quả thu được đã được kiểm chứng bằng cách so sánh nghiệm thu được với nghiệm đã công bố khác. Ảnh hưởng của tham số vật liệu (chỉ số tỉ lệ thể tích), tham số tỷ lệ chiều dài vật liệu, tham số phi địa phương và hệ số của nền đàn hồi đến đáp ứng ổn định của dầm micro được nghiên cứu và thảo luận.

Từ khóa: Vật liệu cơ tính biến thiên; Phân tích ổn định; Độ dốc biến dạng không địa phương; Dầm micro.

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