A method for incremental updating three-way decision with incomplete information system

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ABSTRACT

In recent years, the three-way decisions theory has been developed in both theoretical and practical applications. In fact, data are often incomplete and often change over time. To solve this problem, a method of updating the three-way decisions in the dynamic incomplete information system is proposed. First, we consider the relationship between the change of conditional probabilities for the change of the three regions. Then, we consider the change of conditional probability when objects and attribute values of the object change. From that change, we propose a method for updating three-way decisions for two situations, namely, the objects vary, and the attribute values of an object vary. Finally, we give an example to illustrate this method.

Keywords: Rough Sets; Incomplete Information Systems; Three-way Decisions; Incremental Learning.

1. INTRODUCTION

The rough set theory introduced by Pawlak in 1982 was an effective mathematical tool to analyze the vague description of objects [1]. It provides a method for approximating a target concept according to three pairwise disjoint regions, namely, the positive, boundary and negative regions. These three regions created three-way decision rules of acceptance, rejection, and deferment decisions. These decision rules are made without any error, the classification must be fully correct or certain, leading to limiting their applications in practice. By allowing acceptable tolerance of errors, probabilistic approaches have been applied to rough set theory and are called probabilistic rough set models [2-5]. It used a pair of thresholds α, β on probabilities for defining probabilistic approximations, a generalized probabilistic model is determined, called a decision-theoretic rough set ($DTRS$) [5].

In fact, information systems can evolve over time, that is, some new information becomes available to replace some information that is no longer useful. Because the information table in the rough set consists of data about objects, attributes, and attribute values, the approaches to updating knowledge are mainly discussed on three types, namely, variation of objects, variation of attributes, and variation of attribute values. Therefore, the incremental learning technique is designed for mining dynamic databases. Instead of exploiting the database from scratch, the researchers used the results obtained earlier to facilitate maintenance knowledge in the database was altered. For incremental approaches of updating approximations in rough set theory, the researchers have proposed methods for incremental updating of rough approximations in information systems based on characteristic relations, using lower and upper boundaries, or matrices and so on [6-11].

The three-way decision model is a new way of solving the uncertainty problem that has emerged in recent years. It was suggested by Yao [3-5] and has been used in many real applications, including medical diagnosis, risk investment, face recognition, social networks, etc. The main idea of three-way decisions is to divide a universe, based on the set of criteria, into three disjoint regions called the positive, negative, and boundary regions, respectively. In recent years, many scholars studied three-way decision and its application in a dynamic environment. Yao presented a granular computing perspective on sequential three-way decisions [12]. Li et al. presented a cost-sensitive sequential three-way decision model, which simulates a gradual decision process from rough granules to precise granules [13]. Liu et al. proposed a dynamic DTRS by considering the dynamic change of loss functions under a dynamic decision environment [14]. Luo et al. proposed a method for incremental updating three-way decisions in the incomplete information system when the object set varies through time [15]. Accordingly, the three-way decision rules are updated as new objects are added or removed from a given data set. Xu et al. proposed a stream computing learning method based on existing incremental learning studies to solve challenges resulting from the simultaneous addition and deletion of objects with a complete information system [16]. Luo et al. present the updating properties for dynamic maintenance of approximations when the criteria values in the set-valued decision system evolve with time [10, 17]. Chen et al. proposed a method for dynamic maintenance of decision rules w.r.t attribute values' coarsening and refining [18]. Zeng et al. present fuzzy rough sets approach for incrementally updating approximations in hybrid information systems [19]. Their paper has developed two incremental algorithms to update approximations based on the changing mechanism of attribute values and fuzzy equivalence relations in fuzzy rough sets. Yang et al. proposed a unified dynamic framework of decision theoretic rough sets for incrementally updating three-way probabilistic regions based on a matrix [11]. They introduce a novel matrix approach to updating three-way regions when objects, attributes, and attribute values vary solely or simultaneously. Luo et al. studied the update problem of three-way decisions with a dynamic variation of scales in incomplete multi-scale information systems [20]. These studies have only proposed a method to update three-way decisions in incomplete information systems when only adding or removing an object separately. Research on the method of updating three-way decisions when adding and removing objects simultaneously has only been studied for a complete information system. In this paper, we focus on investigating how to update three-way decision in an incomplete information system for two situations, namely, the objects vary, and the attribute values of an object vary. The contributions of the paper are as follows. We first propose a method to update three-way decisions in incomplete information systems when the conditional probabilities change. This method helps us to update three-way decisions based on existing three-way decisions without having to recalculate from scratch. We then calculate the change of the conditional probability when simultaneously adding and removing objects and when the attribute values of an object change, to serve as a basis for updating three-way decisions in these cases.

This paper is organized as follows: in section 2, some basic concepts of the incomplete decision system and three-way decisions based on rough set theoryare brief reviews. Section 3 gives the principles of incremental updating three-way decisions in the incomplete decision system when the conditional probabilities vary. Section 4 and 5 investigates the tendency to change the conditional probability that an object belongs to a target set. when the objects change and the attribute values of an object change, respectively. Section 6 introduces an illustrative case study. Finally, conclusions and future work are presented in section 7.

2. THEORETICAL BACKGROUND

In this section, we review the basic concepts of a rough set and three-way decisions in an incomplete system.

An information system usually is defined as $\boldsymbol{IIS = (U, A, V, f)}$, where $\boldsymbol{U}$ is a non-empty finite set of objects, $\boldsymbol{A = C \cup D}$ is a non-empty finite set of attributes, where $\boldsymbol{C}$ is the set of condition attribute, $\boldsymbol{D}$ is the set of decision attribute. $\boldsymbol{V = \bigcup_{a \in U} V_a}$ and $\boldsymbol{V_a}$ is a domain of attribute $\boldsymbol{a}$. $\boldsymbol{f: U \times A \to V_a}$ is an information function such that for any $\boldsymbol{a \in A, u \in U, f_a(u) \in V_a}$.

If $\boldsymbol{U}$ contains at least one object with an unknown value, then $\boldsymbol{IIS}$ is called an incomplete information system, otherwise complete [21]. if $\boldsymbol{IIS}$ is an incomplete information system, it may contain several unknown attribute values, but does not include the cases where all objects take the unknown value for an attribute. In the incomplete information system, the unknown values are denoted by special symbol " $*$ ", and they are assumed to belong to the set $\boldsymbol{V_a}$.

Definition 1. Let the incomplete information system $\boldsymbol{IIS = (U;\ A;\ V;\ f); P \subseteq A}$. A tolerance relation [21] $\boldsymbol{TOR_P}$, denotes a binary relation between objects that are possibly equivalent in terms of values of attributes and defined as follows:

$$\boldsymbol{TOR_P = \{(u, v) \in U \times U | \forall a \in P, f_a(u) = f_a(v) \vee f_a(u) = '*' \vee f_a(v) = '*'\}}. \quad (1)$$

This relation is reflexive and symmetric but does not need to be transitive.

Let $\boldsymbol{T_P(u) = \{v \in U | (u, v) \in TOR_P\}}$ be the set of objects which are in relation with $\boldsymbol{x}$ in terms of $\boldsymbol{P}$ in the sense of the above tolerance relation. The lower and upper approximations of target set $\boldsymbol{X \subseteq U}$ w.r.t $\boldsymbol{P}$ is defined by:

$$\boldsymbol{\underline{apr}_P X = \{u \in U | T_P(u) \subseteq X\} = \{u \in X | T_P(u) \subseteq X\}} \quad (2)$$

$$\boldsymbol{\overline{apr}_P X = \{u \in U | T_P(u) \cap X \neq \emptyset\} = \bigcup_{u \in X} T_P(u)} \quad (3)$$

Definition 2. Given a subset of the universe the conditional probability of $\boldsymbol{x}$ belonging to $\boldsymbol{B}$ [22] can be simply estimated as follows:

$$\boldsymbol{Pr(B | T_P(x)) = \frac{|B \cap T_P(x)|}{|T_P(x)|}} \quad (4)$$

Where $|.|$ denotes the cardinality of a set.

By using a pair of thresholds on probability Yao et al. defined three regions, namely, the positive, boundary and negative regions as follows:

Definition 3. [23] Given a pair $\boldsymbol{(\alpha, \beta)}$ of thresholds with $\boldsymbol{\alpha > \beta}$, the $\boldsymbol{(\alpha, \beta)}$- probabilistic positive, boundary and negative regions are defined as follows:

$$\boldsymbol{POS_{(\alpha,\beta)}(X) = \{x \in U | Pr(X | T_P(x)) \geq \alpha\}},$$
$$\boldsymbol{BND_{(\alpha,\beta)}(X) = \{x \in U | \beta < Pr(X | T_P(x)) < \alpha\}},$$
$$\boldsymbol{NEG_{(\alpha,\beta)}(X) = \{x \in U | Pr(X | T_P(x)) \leq \beta\}}. \quad (5)$$

The $\boldsymbol{(\alpha, \beta)}$- Probabilistic lower and upper approximations of $\boldsymbol{C}$ are defined by:

$$\underline{apr}_{(\alpha,\beta)}(X) = \{x \in U | Pr(X|T_P(x)) \geq \alpha\},$$
$$\overline{apr}_{(\alpha,\beta)}(X) = \{x \in U | Pr(X|T_P(x)) > \beta\}. \quad (6)$$

Assume that $C_i \in U/C$; where U/C is equivalent partition of condition attribute; $D_i \in U/D$, where U/D is equivalent partition of the decision attribute. Three-way decision rules, namely, acceptance, rejection, and deferment decisions, respectively, are extracted:

$$DES_P(C_i \to D_i), \text{ for } C_i \in POS_{(\alpha,\beta)}(D_i);$$
$$DES_B(C_i \to D_i), \text{ for } C_i \in BND_{(\alpha,\beta)}(D_i);$$
$$DES_N(C_i \to D_i), \text{ for } C_i \in NEG_{(\alpha,\beta)}(D_i). \quad (7)$$

Where $DES(.)$ denotes the logic formula defining a set.

3. APPROACHES FOR INCREMENTAL UPDATING THREE-WAY DECISION UNDER THE VARIATION OF THE CONDITIONAL PROBABILITY

The three-way decisions model proposed by Yao [3] has been many applied in a wide variety of applications, such as email filtering [24], investment decision [25], and cluster analysis [26]. For dynamic information systems, studies on updating knowledge can be divided into two aspects. First, use a characteristic matrix, relation matrix and intersection matrix to calculate three-way regions. Then, based on the change of matrices, propose a method for updating three-way regions. The other aspect is based on DTRS model, updating knowledge by considering the change of conditional probability that an object belongs to a target set [9, 15], 16]. Our previous study presented a method to update three-way decision when simultaneously adding and deleting objects [27]. In the following, we examine how the three-way regions change when the conditional probabilities change.

Given an incomplete information system $IIS = (U;\ A;\ V;\ f)$. To describe a dynamic maintenance process from time t to time $t+1$, the conditional probabilities at time t are denoted with the same symbol $Pr(t)$ and those at time $t+1$ with the same symbol $Pr(t+1)$.

At time $t+1$, the conditional probabilities do not change, the three-way regions always remain constant. If it decreases, we must recalculate the conditional probabilities of the equivalence classes involved and compare them with the pair of thresholds (α, β) to determine the updating of the three-way decisions.

Proposition 1. Suppose that at time $t+1$, the conditional probabilities decreased, that is, $Pr(t+1) < Pr(t)$, then

$$POS_{(\alpha,\beta)}(X)^{(t+1)} = \begin{cases} POS_{(\alpha,\beta)}(X)^{(t)} \cup \Delta, & p_1 \\ POS_{(\alpha,\beta)}(X)^{(t)} - \Delta' & p_2 \end{cases} \quad (8)$$

Where p_1: $\Delta = \left\{x \in POS_{(\alpha,\beta)}(X)^{(t)} \middle| x \notin T_{P_i}^{(t)} \wedge x \in T_{P_i}^{(t+1)}, Pr\left(X/T_{P_i}^{(t+1)}\right) \geq \alpha\right\}$,

$$p_2: \Delta' = \left\{x \in POS_{(\alpha,\beta)}(X)^{(t)} \middle| x \in T_{P_i}^{(t)}, Pr\left(X/T_{P_i}^{(t+1)}\right) \geq \alpha\right\}.$$

$$BND_{(\alpha,\beta)}(X)^{(t+1)} = \begin{cases} BND_{(\alpha,\beta)}(X)^{(t)} \cup \Delta_1, & b_1 \\ BND_{(\alpha,\beta)}(X)^{(t)} \cup \Delta_2, & b_2 \\ BND_{(\alpha,\beta)}(X)^{(t)} - \Delta_3 & b_3 \end{cases} \quad (9)$$

Where

$$b_1: \Delta_1 = \left\{ x \in POS_{(\alpha,\beta)}(X)^{(t)} \middle| x \in T_{P_i}^{(t)} \land x \in T_{P_i}^{(t+1)}, \beta < Pr\left(X/T_{P_i}^{(t+1)}\right) < \alpha \right\},$$
$$b_2: \Delta_2 = \left\{ x \in BND_{(\alpha,\beta)}(X)^{(t)} \middle| x \notin T_{P_i}^{(t)} \land x \in T_{P_i}^{(t+1)}, \beta < Pr\left(X/T_{P_i}^{(t+1)}\right) < \alpha \right\},$$
$$b_3: \Delta_3 = \left\{ x \in BND_{(\alpha,\beta)}(X)^{(t)} \middle| x \in T_{P_i}^{(t)}, Pr\left(X/T_{P_i}^{(t+1)}\right) < \beta \right\}.$$

$$NEG_{(\alpha,\beta)}(X)^{(t+1)} = \begin{cases} NEG_{(\alpha,\beta)}(X)^{(t)} \cup \Delta'_1, & n_1 \\ NEG_{(\alpha,\beta)}(X)^{(t)} \cup \Delta'_2, & n_2 \\ NEG_{(\alpha,\beta)}(X)^{(t)} \cup \Delta'_3 & n_3 \end{cases} \tag{10}$$

Where $n_1: \Delta'_1 = \left\{ x \in POS_{(\alpha,\beta)}(X)^{(t)} \middle| x \in T_{P_i}^{(t)} \land x \in T_{P_i}^{(t+1)}, Pr\left(X/T_{P_i}^{(t+1)}\right) \leq \beta \right\},$

$$n_2: \Delta'_2 = \left\{ x \in BND_{(\alpha,\beta)}(X)^{(t)} \middle| x \in T_{P_i}^{(t)}, Pr\left(X/T_{P_i}^{(t+1)}\right) \leq \beta \right\},$$
$$b_3: \Delta_3 = \left\{ x \in NEG_{(\alpha,\beta)}(X)^{(t)} \middle| x \notin T_{P_i}^{(t)} \land x \in T_{P_i}^{(t+1)} \right\}.$$

The same as Proposition 1, when the conditional probabilities increase due to a change in an information system, we also recalculate the conditional probabilities of the equivalence classes involved in updating three-way decisions. It is described in the following proposition:

Proposition 2. Suppose that at time $t+1$, the conditional probabilities increased, that is, $Pr(t+1) > Pr(t)$, then

$$POS_{(\alpha,\beta)}(X)^{(t+1)} = \begin{cases} POS_{(\alpha,\beta)}(X)^{(t)} \cup \Delta, & p_1 \\ POS_{(\alpha,\beta)}(X)^{(t)} \cup \Delta' & p_2 \\ POS_{(\alpha,\beta)}(X)^{(t)} \cup \Delta'' & p_3 \end{cases} \tag{11}$$

Where $p_1: \Delta = \left\{ x \in POS_{(\alpha,\beta)}(X)^{(t)} \middle| x \notin T_{P_i}^{(t)} \land x \in T_{P_i}^{(t+1)} \right\},$

$$p_2: \Delta' = \left\{ x \in BND_{(\alpha,\beta)}(X)^{(t)} \middle| x \in T_{P_i}^{(t)}, Pr\left(X/T_{P_i}^{(t+1)}\right) \geq \alpha \right\}.$$
$$p_3: \Delta'' = \left\{ x \in NEG_{(\alpha,\beta)}(X)^{(t)} \middle| x \in T_{P_i}^{(t)}, Pr\left(X/T_{P_i}^{(t+1)}\right) \geq \alpha \right\}.$$

$$BND_{(\alpha,\beta)}(X)^{(t+1)} = \begin{cases} BND_{(\alpha,\beta)}(X)^{(t)} - \Delta_1, & b_1 \\ BND_{(\alpha,\beta)}(X)^{(t)} \cup \Delta_2 & b_2 \\ BND_{(\alpha,\beta)}(X)^{(t)} \cup \Delta_3 & b_3 \end{cases} \tag{12}$$

Where

$$b_1: \Delta_1 = \left\{ x \in BND_{(\alpha,\beta)}(X)^{(t)} \middle| x \in T_{P_i}^{(t)}, Pr\left(X/T_{P_i}^{(t+1)}\right) \geq \alpha \right\},$$
$$b_2: \Delta_2 = \left\{ x \in BND_{(\alpha,\beta)}(X)^{(t)} \middle| x \notin T_{P_i}^{(t)} \land x \in T_{P_i}^{(t+1)}, \beta < Pr\left(X/T_{P_i}^{(t+1)}\right) < \alpha \right\}.$$
$$b_3: \Delta_3 = \left\{ x \in NEG_{(\alpha,\beta)}(X)^{(t)} \middle| x \in T_{P_i}^{(t+1)}, \beta < Pr\left(X/T_{P_i}^{(t+1)}\right) < \alpha \right\}.$$

$$NEG_{(\alpha,\beta)}(X)^{(t+1)} = \begin{cases} NEG_{(\alpha,\beta)}(X)^{(t)} - \Delta'_1, & n_1 \\ NEG_{(\alpha,\beta)}(X)^{(t)} - \Delta'_2 & n_2 \\ NEG_{(\alpha,\beta)}(X)^{(t)} \cup \Delta'_3 & n_3 \end{cases} \tag{13}$$

Where

$$n_1: \Delta'_1 = \left\{x \in NEG_{(\alpha,\beta)}(X)^{(t)} \middle| x \in T_{P_i}^{(t)}, Pr\left(X/T_{P_i}^{(t+1)}\right) \geq \alpha\right\},$$

$$n_2: \Delta'_2 = \left\{x \in NEG_{(\alpha,\beta)}(X)^{(t)} \middle| x \in T_{P_i}^{(t+1)}, \beta < Pr\left(X/T_{P_i}^{(t+1)}\right) < \alpha\right\}.$$

$$b_3: \Delta_3 = \left\{x \in NEG_{(\alpha,\beta)}(X)^{(t)} \middle| x \notin T_{P_i}^{(t)} \wedge x \in T_{P_i}^{(t+1)}, Pr\left(X/T_{P_i}^{(t+1)}\right) \leq \beta\right\}.$$

4. THE VARIATION OF THE CONDITIONAL PROBABILITY WHEN THE OBJECTS VARY

This section proposes a method for updating conditional probabilities when simultaneously adding and deleting objects.

Consider an incomplete information system $IIS = (U, C \cup D, V, f)$. We describe this information system at time t, when the objects have not changed, is $IIS^{(t)} = \left(U^{(t)}, C^{(t)} \cup D^{(t)}, V, f\right)$. And $IIS^{(t+1)} = \left(U^{(t+1)}, C^{(t+1)} \cup D^{(t+1)}, V, f\right)$ denotes the IIS at time $t+1$, when adding object $\overline{x}$ and deleting object $\underline{x}$ simultaneously.

Let $U^{(t)}/TOL_P^{(t)}$, $U^{(t)}/D^{(t)}$ be a set of Tolerance classes formed by $U^{(t)}/TOL_P^{(t)} = \left\{T_{P_1}^{(t)}, \ldots, T_{P_m}^{(t)}\right\}$ and $U^{(t)}/D^{(t)} = \left\{D_1^{(t)}, \ldots, D_n^{(t)}\right\}$, respectively, at time t. And those at time $t+1$ be $U^{(t+1)}/TOL_P^{(t+1)} = \left\{T_{P_1}^{(t+1)}, \ldots, T_{P_m}^{(t+1)}\right\}$ and $U^{(t+1)}/D^{(t+1)} = \left\{D_1^{(t+1)}, \ldots, D_n^{(t+1)}\right\}$.

Where $\left\{T_{P_i}^{(t)}\right\}, \left\{T_{P_i}^{(t+1)}\right\}, i = 1, 2, ..m$ are tolerance classes at time t and $t+1$, respectively.

When the dynamic information system is simultaneously adding and deleting objects, the equivalent partition of Tolerance relation and decision are updated as follows:

Proposition 3. Given an information system at time t is $IIS^{(t)} = \left(U^{(t)}, C^{(t)} \cup D^{(t)}, V, f\right)$. We have

$$U^{(t+1)}/TOL_P^{(t+1)} = \begin{cases} \left\{T_{P_1}^{(t+1)}, \ldots, T_{P_m}^{(t+1)}\right\}, \textbf{if } \exists x U, a \in P; \left(f_a(x) = f_a(\overline{x})\right) \vee (f_a(x) = *) \vee (f_a(\overline{x}) = *) \\ \left\{T_{P_1}^{(t+1)}, \ldots, T_{P_m}^{(t+1)}, T_{P_{m+1}}^{(t+1)}\right\}, \textbf{otherwise}. \end{cases}$$

$$\text{Where } T_{P_m}^{(t+1)} = \begin{cases} T_{P_i}^{(t+1)} - \{\underline{x}\}, \textbf{if } \underline{x} \in T_{P_i}^{(t)} \wedge \overline{x} \notin T_{P_i}^{(t)}, \\ T_{P_i}^{(t+1)} \cup \{\overline{x}\}, \textbf{if } \underline{x} \notin T_{P_i}^{(t)} \wedge \overline{x} \in T_{P_i}^{(t)}, \\ T_{P_i}^{(t+1)} \cup \{\overline{x}\} - \{\underline{x}\}, \textbf{if } \underline{x} \in T_{P_i}^{(t)} \wedge \overline{x} \in T_{P_i}^{(t)}, \\ T_{P_i}^{(t)}, \quad \textbf{otherwise}. \end{cases}$$

For $1 \leq i \leq m$ and $T_{P_{m+1}}^{(t+1)} = \{\overline{x}\}$ for $i = m+1$.

$$U^{(t+1)}/D^{(t+1)} = \begin{cases} \left\{D_1^{(t+1)}, \ldots, D_n^{(t+1)}\right\}, \textbf{if } \exists x \in U, \forall d \in D; \left(d(x) = d(\overline{x})\right), \\ \left\{D_1^{(t+1)}, \ldots, D_n^{(t+1)}, D_{n+1}^{(t+1)}\right\}, \quad \textbf{otherwise}. \end{cases}$$

Where $D_j^{(t+1)} = \begin{cases} D_j^{(t)} - \{\underline{x}\}, \text{if } \underline{x} \in D_j^{(t)} \wedge \overline{x} \notin D_j^{(t)}, \\ D_j^{(t)} \cup \{\overline{x}\}, \text{if } \underline{x} \notin D_j^{(t)} \wedge \overline{x} \in D_j^{(t)} \\ D_j^{(t)} \cup \{\overline{x}\} - \{\underline{x}\}, \text{if } \underline{x} \in D_j^{(t)} \wedge \overline{x} \in D_j^{(t)} \\ D_j^{(t)}, \quad \text{otherwise}. \end{cases}$

For $1 \le j \le n$ and $D_j^{(t+1)} = \{\overline{x}\}$ for $j = n + 1$.

In the above equation, we see that there is a case where new object $\overline{x}$ does not belongs to any existing equivalence class of condition or any existing equivalence class of decision. In this case, $\overline{x}$ will form a new class, respectively.

When simultaneously adding and deleting objects, the trends of conditional probabilities be varied as follows:

Theorem 4. Let $IIS^{(t)} = (U^{(t)}, C^{(t)} \cup D^{(t)}, V, f)$ and $IIS^{(t+1)} = (U^{(t+1)}, C^{(t+1)} \cup D^{(t+1)}, V, f)$ be an information system at t and $t + 1$, respectively. Assuming there is an object $\overline{x}$ is added to IIS while $\underline{x}$ is deleted from IIS and

Patterns	Trend
(i) $(\overline{x} \notin T_{P_i}^{(t+1)} \wedge \overline{x} \notin D_j^{(t+1)}) \wedge (\underline{x} \notin T_{P_i}^{(t+1)} \wedge \underline{x} \notin D_j^{(t+1)})$; (ii) $(\overline{x} \notin T_{P_i}^{(t+1)} \wedge \overline{x} \notin D_j^{(t+1)}) \wedge (\underline{x} \notin T_{P_i}^{(t+1)} \wedge \underline{x} \in D_j^{(t+1)})$; (iii) $(\overline{x} \notin T_{P_i}^{(t+1)} \wedge \overline{x} \in D_j^{(t+1)}) \wedge (\underline{x} \notin T_{P_i}^{(t+1)} \wedge \underline{x} \notin D_j^{(t+1)})$; (iv) $(\overline{x} \notin T_{P_i}^{(t+1)} \wedge \overline{x} \in D_j^{(t+1)}) \wedge (\underline{x} \notin T_{P_i}^{(t+1)} \wedge \underline{x} \in D_j^{(t+1)})$; (v) $(\overline{x} \in T_{P_i}^{(t+1)} \wedge \overline{x} \notin D_j^{(t+1)}) \wedge (\underline{x} \in T_{P_i}^{(t+1)} \wedge \underline{x} \notin D_j^{(t+1)})$; (vi) $(\overline{x} \in T_{P_i}^{(t+1)} \wedge \overline{x} \in D_j^{(t+1)}) \wedge (\underline{x} \in T_{P_i}^{(t+1)} \wedge \underline{x} \in D_j^{(t+1)})$;	$Pr(t+1) = Pt(t)$
(vii) $(\overline{x} \notin T_{P_i}^{(t+1)} \wedge \overline{x} \notin D_j^{(t+1)}) \wedge (\underline{x} \in T_{P_i}^{(t+1)} \wedge \underline{x} \notin D_j^{(t+1)})$; $(viii)$ $(\overline{x} \notin T_{P_i}^{(t+1)} \wedge \overline{x} \in D_j^{(t+1)}) \wedge (\underline{x} \in T_{P_i}^{(t+1)} \wedge \underline{x} \notin D_j^{(t+1)})$; (ix) $(\overline{x} \in T_{P_i}^{(t+1)} \wedge \overline{x} \in D_j^{(t+1)}) \wedge (\underline{x} \notin T_{P_i}^{(t+1)} \wedge \underline{x} \notin D_j^{(t+1)})$; (x) $(\overline{x} \in T_{P_i}^{(t+1)} \wedge \overline{x} \in D_j^{(t+1)}) \wedge (\underline{x} \in T_{P_i}^{(t+1)} \wedge \underline{x} \notin D_j^{(t+1)})$; (xi) $(\overline{x} \in T_{P_i}^{(t+1)} \wedge \overline{x} \in D_j^{(t+1)}) \wedge (\underline{x} \notin T_{P_i}^{(t+1)} \wedge \underline{x} \in D_j^{(t+1)})$;	$Pr(t+1) > Pt(t)$
(xii) $(\overline{x} \notin T_{P_i}^{(t+1)} \wedge \overline{x} \notin D_j^{(t+1)}) \wedge (\underline{x} \in T_{P_i}^{(t+1)} \wedge \underline{x} \notin D_j^{(t+1)})$; $(xiii)$ $(\overline{x} \notin T_{P_i}^{(t+1)} \wedge \overline{x} \in D_j^{(t+1)}) \wedge (\underline{x} \in T_{P_i}^{(t+1)} \wedge \underline{x} \notin D_j^{(t+1)})$; (xiv) $(\overline{x} \in T_{P_i}^{(t+1)} \wedge \overline{x} \in D_j^{(t+1)}) \wedge (\underline{x} \notin T_{P_i}^{(t+1)} \wedge \underline{x} \notin D_j^{(t+1)})$; (xv) $(\overline{x} \in T_{P_i}^{(t+1)} \wedge \overline{x} \in D_j^{(t+1)}) \wedge (\underline{x} \in T_{P_i}^{(t+1)} \wedge \underline{x} \notin D_j^{(t+1)})$; (xvi) $(\overline{x} \in T_{P_i}^{(t+1)} \wedge \overline{x} \in D_j^{(t+1)}) \wedge (\underline{x} \notin T_{P_i}^{(t+1)} \wedge \underline{x} \in D_j^{(t+1)})$;	$Pr(t+1) < Pt(t)$

Proof.

(i) Since $\left(\overline{x} \notin T_{P_i}^{(t+1)} \wedge \overline{x} \notin D_j^{(t+1)}\right) \wedge \left(\underline{x} \notin T_{P_i}^{(t+1)} \wedge \underline{x} \notin D_j^{(t+1)}\right)$ then $\left(T_{P_i}^{(t+1)} = T_{P_i}^{(t)}\right) \wedge \left(D_j^{(t+1)} = D_j^{(t)}\right)$.

$\Rightarrow Pr(t+1) = Pr\left(D_j^{(t+1)} / T_{P_i}^{(t+1)}\right) = \frac{\left|D_j^{(t+1)} \cap T_{P_i}^{(t+1)}\right|}{\left|T_{P_i}^{(t+1)}\right|} = \frac{\left|D_j^{(t)} \cap T_{P_i}^{(t)}\right|}{\left|T_{P_i}^{(t)}\right|} = Pr\left(D_j^{(t)} / T_{P_i}^{(t)}\right) = Pt(t)$.

Like $(\boldsymbol{i})$ we have proofs of $(\boldsymbol{ii})$; $(\boldsymbol{iii})$; $(\boldsymbol{iv})$; $(\boldsymbol{v})$; $(\boldsymbol{vi})$.

With the assumptions $(\boldsymbol{vii})$ to $(\boldsymbol{viii})$ above, we see that

$$\left|\boldsymbol{D}_{\boldsymbol{j}}^{(\boldsymbol{t}+\boldsymbol{1})} \cap \boldsymbol{T}_{\boldsymbol{P}_{\boldsymbol{i}}}^{(\boldsymbol{t}+\boldsymbol{1})}\right| = \left|\boldsymbol{D}_{\boldsymbol{j}}^{(\boldsymbol{t})} \cap \boldsymbol{T}_{\boldsymbol{P}_{\boldsymbol{i}}}^{(\boldsymbol{t})}\right| \text{ and } \left|\boldsymbol{T}_{\boldsymbol{P}_{\boldsymbol{i}}}^{(\boldsymbol{t}+\boldsymbol{1})}\right| < \left|\boldsymbol{T}_{\boldsymbol{P}_{\boldsymbol{i}}}^{(\boldsymbol{t})}\right|.$$

With the assumptions $(\boldsymbol{ix})$ to $(\boldsymbol{xi})$, implies that

$$\left|\boldsymbol{D}_{\boldsymbol{j}}^{(\boldsymbol{t}+\boldsymbol{1})} \cap \boldsymbol{T}_{\boldsymbol{P}_{\boldsymbol{i}}}^{(\boldsymbol{t}+\boldsymbol{1})}\right| > \left|\boldsymbol{D}_{\boldsymbol{j}}^{(\boldsymbol{t})} \cap \boldsymbol{T}_{\boldsymbol{P}_{\boldsymbol{i}}}^{(\boldsymbol{t})}\right| \text{ and } \left|\boldsymbol{T}_{\boldsymbol{P}_{\boldsymbol{i}}}^{(\boldsymbol{t}+\boldsymbol{1})}\right| > \left|\boldsymbol{T}_{\boldsymbol{P}_{\boldsymbol{i}}}^{(\boldsymbol{t})}\right|$$

$$\Rightarrow \frac{\left|\boldsymbol{D}_{\boldsymbol{j}}^{(\boldsymbol{t}+\boldsymbol{1})} \cap \boldsymbol{T}_{\boldsymbol{P}_{\boldsymbol{i}}}^{(\boldsymbol{t}+\boldsymbol{1})}\right|}{\left|\boldsymbol{T}_{\boldsymbol{P}_{\boldsymbol{i}}}^{(\boldsymbol{t}+\boldsymbol{1})}\right|} > \frac{\left|\boldsymbol{D}_{\boldsymbol{j}}^{(\boldsymbol{t})} \cap \boldsymbol{T}_{\boldsymbol{P}_{\boldsymbol{i}}}^{(\boldsymbol{t})}\right|}{\left|\boldsymbol{T}_{\boldsymbol{P}_{\boldsymbol{i}}}^{(\boldsymbol{t})}\right|} \Rightarrow \boldsymbol{Pr}(\boldsymbol{t}+\boldsymbol{1}) > \boldsymbol{Pt}(\boldsymbol{t}).$$

Similar to above we have proofs of $(\boldsymbol{xii})$ to $(\boldsymbol{xvi})$,

Another type of dynamic information system is a variation of attribute values, composite, variation of condition attribute values and variation of decision attribute values, solely or simultaneously. When an object in the information system changes the attribute value, we can consider it the same as deleting the object with the old information and adding the object with new information simultaneously. According to this approach, we need to investigate all changing situations of both tolerance classes and decision classes, to calculate the change of conditional probabilities. However, when the object only changes the value of the conditional attribute or decision attribute, we only need to consider the individual fluctuations of the tolerance classes, or decision classes instead of having to consider both. Even when both the conditional and the decisive attribute values of an object change simultaneously, the investigation of equivalence classes is also simpler because this object certainly does not belong to the equivalence classes where it belongs before. This method helps shorten the time to investigate the available knowledge, making the update time of decision rules shorter.

5. THE VARIATION OF THE CONDITIONAL PROBABILITY WHEN THE ATTRIBUTE VALUES OF OBJECTS VARY

In this section, we focus on studying the change of conditional probability when the attribute values of an object vary over time. First, we consider the case in which the conditional attribute value and the decision attribute value of an object change simultaneously. Then, we look at when they change individually.

At time t let $U^{(t)}/TOL_P^{(t)} = \left\{T_{Pi}^{(t)}\right\}_{i=1,2,..,n}, U^{(t)}/D^{(t)} = \left\{D_j^{(t)}\right\}_{j=1,2,..,m}$ be tolerance classes and decision classes, respectively. At time t+1, when $x_k \in U$, the tolerance classes and the decision classes are updated as follows:

$$U^{(t)}/TOL_P^{(t)} = \begin{cases} \left\{T_{Pi}^{(t)}\right\}_{i=1,..,n}, if\ \exists x \in U, \forall a \in P; \Big(\big(f_a(x) = f_a(x_k)\big) \\ \qquad\qquad \vee (f_a(x) = *) \vee (f_a(x_k) = *)\Big), \\ \left\{T_{Pi}^{(t)}\right\}_{i=1,..,n+1}, \text{otherwise.} \end{cases} \quad (14)$$

Where $T_{Pi}^{(t+1)}(x) = \begin{cases} T_{Pi}^{(t)} - \{x_k\}, & x_k \in T_{Pi}^{(t)} \wedge x_k \notin T_{Pi}^{(t+1)}, 1 \le i \le m, \\ T_{Pi}^{(t)} \cup \{x_k\}, & x_k \notin T_{Pi}^{(t)} \wedge x_k \in T_{Pi}^{(t+1)}, 1 \le i \le m, \\ T_{Pi}^{(t)}, & otherwise, \end{cases}$

And $U^{(t+1)}/D^{(t+1)} = \begin{cases} \{D_j^{t+1}\}_{j=1,2,..,m}, & if\ \exists x \in U, \forall d \in D, (d(x) = d(x_k)), \\ \{D_j^{t+1}\}_{j=1,2,..,m+1}, & otherwise. \end{cases}$

Where $D_j^{(t+1)} = \begin{cases} D_j^{(t)} - \{x_k\}, & x_i \in D_j^{(t)} \wedge x_i \notin D_j^{(t+1)}, 1 \le j \le n, \\ D_j^{(t)} \cup \{x_k\}, & x_i \notin D_j^{(t)} \wedge x_i \in D_j^{(t+1)}, 1 \le j \le n, \\ D_j^{(t)}, & otherwise. \end{cases}$

When the object x_k varies a condition and decision value simultaneously, there is a case where new attribute value of object x_i does not belong to any existing tolerance class or any existing decision class. In this case, x_i will form a new class, respectively and we can be directly determined three-way decisions.

Since both the condition and decision attribute values vary, the tolerance classes and the decision classes are also changed. In this case, to calculate the changing trend of conditional probability, we must consider the change in both. The following theorems will thoroughly analyze this issue.

Theorem 5. Let $IIS = (U, A = C \cup D, V, f)$ be an information system, where $U = \{x_1, x_2, .., x_n\}$, $P \subseteq A$. Suppose $\exists x_k \in U, a_l \in C$ and $d \in D$, in order to $a_l(x_k)^{(t+1)} \neq a_l(x_k)^{(t)} \wedge d(x_k)^{(t+1)} \neq d(x_k)^{(t)}$.

Patterns	Trend
$(x_k \notin T_{Pi}^{(t)} \wedge x_k \notin D_j^{(t)}) \wedge (x_k \notin T_{Pi}^{(t+1)} \wedge x_k \notin D_j^{(t+1)})$ $(x_k \notin T_{Pi}^{(t)} \wedge x_k \in D_j^{(t)}) \wedge (x_k \notin T_{Pi}^{(t+1)} \wedge x_k \notin D_j^{(t+1)})$ $(x_k \notin T_{Pi}^{(t)} \wedge x_k \notin D_j^{(t)}) \wedge (x_k \notin T_{Pi}^{(t+1)} \wedge x_k \in D_j^{(t+1)})$	$Pr(D_j^{(t+1)}/T_{Pi}) = Pr(D_j^{(t)}/T_{Pi})$
$(x_k \in T_{Pi}^{(t)} \wedge x_k \in D_j^{(t)}) \wedge (x_k \notin T_{Pi}^{(t+1)} \wedge x_k \notin D_j^{(t+1)})$ $(x_k \notin T_{Pi}^{(t)} \wedge x_k \notin D_j^{(t)}) \wedge (x_k \in T_{Pi}^{(t+1)} \wedge x_k \notin D_j^{(t+1)})$ $(x_k \notin T_{Pi}^{(t)} \wedge x_k \in D_j^{(t)}) \wedge (x_k \in T_{Pi}^{(t+1)} \wedge x_k \notin D_j^{(t+1)})$	$Pr(D_j^{(t+1)}/T_{Pi}) < Pr(D_j^{(t)}/T_{Pi})$
$(x_k \in T_{Pi}^{(t)} \wedge x_k \notin D_j^{(t)}) \wedge (x_k \notin T_{Pi}^{(t+1)} \wedge x_k \notin D_j^{(t+1)})$ $(x_k \in T_{Pi}^{(t)} \wedge x_k \notin D_j^{(t)}) \wedge (x_k \notin T_{Pi}^{(t+1)} \wedge x_k \in D_j^{(t+1)})$ $(x_k \notin T_{Pi}^{(t)} \wedge x_k \notin D_j^{(t)}) \wedge (x_k \in T_{Pi}^{(t+1)} \wedge x_k \in D_j^{(t+1)})$	$Pr(D_j^{(t+1)}/T_{Pi}) > Pr(D_j^{(t)}/T_{Pi})$

Proof: This proof is similar to the proof of Theorem 4.■

If at time $\boldsymbol{t+1}$, only the conditional attribute value on an attribute of object $\boldsymbol{x_k \in U}$ varies then we have $\boldsymbol{D_j^{(t+1)} = D_j^{(t)}}$. In this case, the changing trend of the conditional probability is shown in the following table:

Patterns	Trend
$\boldsymbol{x_k \notin T_{P_i}^{(t)} \wedge x_k \notin T_{P_i}^{(t+1)}}$	$\boldsymbol{Pr(X/T_{P_i}^{(t+1)}) = Pr(X/T_{P_i}^{(t)})}$
$\boldsymbol{x_k \in T_{P_i}^{(t)} \wedge x_k \notin T_{P_i}^{(t+1)} \wedge x_k \in X}$ $\boldsymbol{x_k \notin T_{P_i}^{(t)} \wedge x_k \in T_{P_i}^{(t+1)} \wedge x_k \notin X}$	$\boldsymbol{Pr(X/T_{P_i}^{(t+1)}) < Pr(X/T_{P_i}^{(t)})}$
$\boldsymbol{x_k \in T_{P_i}^{(t)} \wedge x_k \notin T_{P_i}^{(t+1)} \wedge x_k \notin X}$ $\boldsymbol{x_k \notin T_{P_i}^{(t)} \wedge x_k \in T_{P_i}^{(t+1)} \wedge x_k \in X}$	$\boldsymbol{Pr(X/T_{P_i}^{(t+1)}) > Pr(X/T_{P_i}^{(t)})}$

If at time $\boldsymbol{t+1}$, only the decision attribute value on an attribute of object $\boldsymbol{x_k} \in \boldsymbol{U}$ varies then we have $\boldsymbol{T_{P_i}^{(t+1)}} = \boldsymbol{T_{P_i}^{(t)}}$. Therefore, the changing trend of the conditional probability is shown in the following table:

Patterns	Trend
$\boldsymbol{x_k \in D_j^{(t)} \wedge x_k \notin D_j^{(t+1)} \wedge x_k \notin T_{P_i}}$ $\boldsymbol{x_k \notin D_j^{(t)} \wedge x_k \in D_j^{(t+1)} \wedge x_k \notin T_{P_i}}$ $\boldsymbol{x_k \notin D_j^{(t)} \wedge x_k \notin D_j^{(t+1)}}$	$\boldsymbol{Pr\left(D_j^{(t+1)}/T_{P_i}\right) = Pr\left(D_j^{(t)}/T_{P_i}\right)}$
$\boldsymbol{x_k \notin D_j^{(t)} \wedge x_k \in D_j^{(t+1)} \wedge x_k \in T_{P_i}}$	$\boldsymbol{Pr\left(D_j^{(t+1)}/T_{P_i}\right) > Pr\left(D_j^{(t)}/T_{P_i}\right)}$
$\boldsymbol{x_k \in D_j^{(t)} \wedge x_k \notin D_j^{(t+1)} \wedge x_k \in T_{P_i}}$	$\boldsymbol{Pr\left(D_j^{(t+1)}/T_{P_i}\right) < Pr\left(D_j^{(t)}/T_{P_i}\right)}$

6. AN ILLUSTRATIVE CASE STUDY

In the above section, we proposed a method for updating three-way decision when the attribute values of an object vary. In the following, by means of the analysis of an example, we explain how we can use this method. In this section, we only consider the case when both the condition attribute and the decision attribute change simultaneously. Other cases are considered similarly.

First step, we are calculating the conditional probability to induce three-way decision regions.

Given IIS at the time t shown in table 1.

Table 1. *The information system at time t.*

ar	a_1	a_2	a_3	a_4	d
1	Low	High	Full	High	Excel
2	Medium	Medium	Full	Low	Excel
3	Low	Medium	Medium	*	Poor
4	Low	*	*	High	Poor
5	High	Low	Full	High	Good
6	High	*	Full	High	Good
7	High	Low	Full	High	Poor
8	High	Low	Full	High	Good

Table 2. *The information system at time t +1.*

Car	a_1	a_2	a_3	a_4	d
1	Low	High	Full	High	Excel
2	Medium	Medium	Full	Low	Excel
3	Low	Medium	Medium	*	Poor
4	Low	*	*	High	Poor
5	High	Low	Full	High	Good
6	Low	*	Full	High	Excel
7	High	Low	Full	High	Poor
8	High	Low	Full	High	Good

et $C = \{a_1, a_2, a_3, a_4\}$

From table 1 we can calculate the tolerance classes and decision classes as follows:

$T_C(1) = \{1\}; T_C(2) = \{2\}; T_C(3) = T_C(4) = \{3,4\}, T_C(5) = T_C(6) = T_C(7) = T_C(8) = \{5,6,7,8\}$.

From there we get the partition and the conditional probabilities:

- $U^{(t)}|TOL_C^{(t)} = \left\{T_{C1}^{(t)}, T_{C2}^{(t)}, T_{C3}^{(t)}, T_{C4}^{(t)}\right\}$; where $T_{C1}^{(t)} = \{1\}; T_{C2}^{(t)} = \{2\}; T_{C3}^{(t)} = \{3,4\}; T_{C4}^{(t)} = \{5,6,7,8\}$.
- $U^{(t)}|D^{(t)} = \left\{D_1^{(t)}, D_2^{(t)}, D_3^{(t)}\right\}$; where $D_1^{(t)} = \{1,2\}; D_2^{(t)} = \{3,4,7\}; D_3^{(t)} = \{5,6,8\}$.

With a pair of thresholds: (0,8; 0,6), we induce the three-way decisions of $D1$; $D2$; $D3$ (follow definition 3).

$$D_1^{(t)}:\begin{cases}POS_{(\alpha,\beta)}\left(D_1^{(t)}\right)=T_{C1}^{(t)}\cup T_{C2}^{(t)}\\ BND_{(\alpha,\beta)}\left(D_1^{(t)}\right)=\emptyset\\ NEG_{(\alpha,\beta)}\left(D_1^{(t)}\right)=T_{C3}^{(t)}\cup T_{C4}^{(t)}\end{cases}\quad D_2^{(t)}:\begin{cases}POS_{(\alpha,\beta)}\left(D_2^{(t)}\right)=T_{C3}^{(t)}\\ BND_{(\alpha,\beta)}\left(D_1^{(t)}\right)=\emptyset\\ NEG_{(\alpha,\beta)}\left(D_1^{(t)}\right)=T_{C1}^{(t)}\cup T_{C2}^{(t)}\cup T_{C4}^{(t)}\end{cases}$$

$$D_3^{(t)}:\begin{cases}POS_{(\alpha,\beta)}\left(D_3^{(t)}\right)=\emptyset\\ BND_{(\alpha,\beta)}\left(D_3^{(t)}\right)=T_{C4}^{(t)}\\ NEG_{(\alpha,\beta)}\left(D_3^{(t)}\right)=T_{C1}^{(t)}\cup T_{C2}^{(t)}\cup T_{C3}^{(t)}\end{cases}$$

Next, we consider the case of the attribute values of object 6 varying simultaneously. Suppose at time $t+1$, the attribute values $a_1(6)=\{low\}\wedge d(6)=\{Excel\}$ as shown in table 2.

Like the above cases, we can calculate the tolerance classes, decision classes and conditional probabilities as follows:

$T_{C1}^{(t+1)}=T_{C1}^{(t)}-\{6\}=\{1,6\}$; $T_{C4}^{(t+1)}=T_{C4}^{(t)}-\{6\}=\{5,7,8\}$; $D_1^{(t+1)}=D_1^{(t)}\cup\{6\}=\{1,2,6\}$; $D_3^{(t+1)}=D_3^{(t)}-\{6\}=\{5,8\}$.

$D_1^{(t+1)}$: $Pr\left(D_1^{(t+1)}|T_{C1}^{(t+1)}\right)=1$; $Pr\left(D_1^{(t+1)}|T_{C4}^{(t+1)}\right)=0$.

$D_2^{(t+1)}$: $Pr\left(D_2^{(t+1)}|T_{C1}^{(t+1)}\right)=0$; $Pr\left(D_1^{(t+1)}|T_{C4}^{(t+1)}\right)=\frac{1}{3}$.

$D_3^{(t+1)}$: $Pr\left(D_3^{(t+1)}|T_{C1}^{(t+1)}\right)=0$; $Pr\left(D_3^{(t+1)}|T_{C4}^{(t+1)}\right)=\frac{2}{3}$.

By those results, we can obtain:

$$D_1^{(t+1)}:\begin{cases}POS_{(\alpha,\beta)}\left(D_1^{(t+1)}\right)=T_{C1}^{(t+1)}\cup T_{C2}^{(t+1)},\\ BND_{(\alpha,\beta)}\left(D_1^{(t+1)}\right)=\emptyset,\\ NEG_{(\alpha,\beta)}\left(D_1^{(t+1)}\right)=T_{C3}^{(t+1)}\cup T_{C4}^{(t+1)}.\end{cases}\quad D_2^{(t+1)}:\begin{cases}POS_{(\alpha,\beta)}\left(D_2^{(t+1)}\right)=T_{C3}^{(t+1)}\\ BND_{(\alpha,\beta)}\left(D_1^{(t+1)}\right)=\emptyset\\ NEG_{(\alpha,\beta)}\left(D_1^{(t+1)}\right)=T_{C1}^{(t+1)}\cup T_{C2}^{(t+1)}\cup T_{C4}^{(t+1)}\end{cases}$$

$$D_3^{(t+1)}:\begin{cases}POS_{(\alpha,\beta)}\left(D_3^{(t+1)}\right)=\emptyset\\ BND_{(\alpha,\beta)}\left(D_3^{(t+1)}\right)=T_{C4}^{(t+1)}\\ NEG_{(\alpha,\beta)}\left(D_3^{(t+1)}\right)=T_{C1}^{(t+1)}\cup T_{C2}^{(t+1)}\cup T_{C3}^{(t)}\end{cases}$$

By means of the analysis of the above example, we explain how we can update the three-way decisions when the value of the condition attribute and decision change simultaneously. At the same time, by this example, we have shown that this method reduces the calculation steps to update three-way decisions for information systems with various attribute values.

7. CONCLUSIONS

This paper studies the three-way decision for a dynamic information system. Two strategies for updating three-way decisions are proposed with the variations of objects and the variations of attribute values of the object, respectively. For this information system, previous studies have given an approach by matrix approach. Our method is based on the change of conditional probabilities to update three-way decisions. The proposed method can update decision rules by modifying the original three regions without having to recalculate from the beginning. In the information system with the attribute values of the objects vary, we can use the method for updating three regions when the objects vary.

However, if the system only changes the attribute values, we should use the second approach. Because it helps to update three-way decisions faster due to less calculation. Finally, we have given an example to describe how we use this method.

Our future work will focus on experimentation, evaluation, and comparison in real databases to validate the proposed approach.

REFERENCES

[1]. P. Z., "*Rough Sets*," International Journal of Computer and Information Sciences, vol. 11, pp. 341–356 (1981).

[2]. Yao. Y. Y, "*Probabilistic rough set approximations*," International Journal of Approximation Reasoning 49, pp. 255–271 (2008).

[3]. Yao. Y. Y, "*Three-way decisions with probabilistic rough set*," Information Sciences 180, pp. 341–353 (2010).

[4]. Yao. Y. Y, "*Three-way decisions with probabilistic rough set*," Information Sciences 181, pp. 1080–1096, (2011).

[5]. Yao. Y. Y, "*An outline of a theory of three-way decisions*," in Yao, J., Yang, Y., Slowinski, R., Greco, S., Li, H., Mitra, S., Polkowski, L. (eds.) RSCTC 2012. LNCS (LNAI), vol. 7413, pp. 1–17 (2012).

[6]. T. R. Li, D. Ruan, W. Geert, J. Song and Y. Xu, "*A rough sets based characteristic relation approach for dynamic attribute generalization in data mining*," KnowledgeBased Systems 20, pp. 485–494 (2007).

[7]. Chien-Chung Chart, "*A rough set approach to attribute generalization in data mining*," Journal of Information Sciences 107, pp. 169–176 (1998).

[8]. H. M. Chen, T. R. Li, D. Ruan, J. H. Lin and C. X. Hu, "*Threeway decisions in dynamic decision theoretic rough sets*," in IEEE Transactions on Knowledge and Data Engineering, vol. 25, no. 2. Springer, Heidelberg, pp. 274–284 (2013).

[9]. C. Luo, T. R. Li and H. M. Chen, "*Dynamic Maintenance of Three-Way Decision Rules*," in rough set and knowledge technology, pp. 801–811 (2014).

[10]. C. Luo, T. R. Li and H. M. Chen, "*Dynamic maintenance of approximations in set-valued ordered decision systems under the attribute generalization*," in Information Sciences 257, pp. 210–228 (2014).

[11]. Xin Yang, Tianrui Li, Dun Liu, Hongmei Chen and C. Luo, "*A unified framework of dynamic three-way probabilistic rough sets*," Information Sciences (2017).

[12]. Yao. Y. Y, "*Granular computing and sequential three-way decisions*," in Lingras, P., Wolski, M., Cornelis, C., Mitra, S., Wasilewski, P. (eds.) RSKT 2013, LNCS, vol. 8171, pp. 16–27 (2013).

[13]. H. Li, X. Zhou, B. Huang, and D. Liu, "*Cost-sensitive threeway decision: A sequential strategy*," in Lingras, P., Wolski, M., Cornelis, C., Mitra, S., Wasilewski, P. (eds.) RSKT 2013, LNCS, vol. 8171. Springer, Heidelberg, pp. 325–337 (2013).

[14]. D. Liu, T. Li, and D. Liang, "*Three-way decisions in dynamic decision theoretic rough sets*," in Lingras, P., Wolski, M., Cornelis, C., Mitra, S., Wasilewski, P. (eds.) RSKT 2013, LNCS, vol. 8171. Springer, Heidelberg, pp. 291–301 (2013).

[15]. C. Luo and T. R. Li, "*Incremental three-way decisions with incomplete information*," in Rough sets and current trends in computing, pp. 128–135 (2014).

[16]. J. Xu, D. Miao, Y.Zhang and Z. Zhang, "*A Three-way Decisions Model with Probabilistic Rough Sets for Stream Computing*," International Journal of Approximate Reasoning (2017).

[17]. C. Luo, T. R. Li, H. Chen and D. Liu, "*Incremental approaches for updating approximations in set-valued ordered information systems*," in Knowledge-Based Systems, vol. 50, pp. 218–223 (2013).

[18]. H. M. Chen, T. R. Li, C. Luo, S. J. Horng and G. Wang, "*A Rough Set-Based Method for Updating Decision Rules on Attribute Values*' Coarsening and Refining," in IEEE Transactions on Knowledge and Data Engineering, vol. 26, no. 12, pp. 2886–2889 (2014).
[19]. A.P. zeng,T. R. Li, J. Hu, H. M. Chen and C. Luo, "*Dynamical updating fuzzy rough approximations for hybrid data under the variation of attribute values*," Information Sciences, pp. 1–26 (2016).
[20]. C. Luo, T. R. Li, Y.Y. Huang and H. Fujita, "*Updating threeway decisions in incomplete multi-scale information systems*," Information Sciences, pp. 1–19 (2018).
[21]. K. M., "*Rough set approach to incomplete information system*," Information Sciences, vol. 112 (1998).
[22]. R. O. Duda and P. E. Hart, "*Pattern Classification and Scene Analysis*," Wiley, New York (1973).
[23]. Y. Y. Yao, "*Rough Sets: neighborhood systems, and granular computing*," in Proceedings of the IEEE Canadian Conference on Electrical and Computer Engineering, pp. 1553–1558 (1999).
[24]. B. Zhou, Y. Y. Yao, and J. Luo, in Farzindar, A., Keˇselj, V. (eds.) Canadian AI. LNCS (LNAI), vol. 6085 (2010).
[25]. D. Liu, Y. Y. Yao, and T. R. Li, "*Three-way investment decisions with Decision theoretic rough sets*," International Journal of Computational Intelligence Systems 4, pp. 66–74 (2011).
[26]. H. Yu, Z. G. Liu, and G. Y. Wang, "*An automatic method to determine the number of clusters using decision-theoretic rough set*," International Journal of Approximate Reasoning 55(1), pp. 101–115 (2014).
[27]. H. Tran, T. Cao, K. Yamada, and D. V. Nguyen, "*Incremental updating methods with three-way decision models in incomplete information systems*," in SCISISIS2018 (2018).

TÓM TẮT

Cập nhật tăng cường các quyết định ba chiều trong hệ thống thông tin không đầy đủ

Trong những năm gần đây, quyết định ba chiều đã phát triển cả về lý thuyết và ứng dụng thực tiễn. Trên thực tế, cơ sở dữ liệu thường không đầy đủ và thường thay đổi theo thời gian. Để giải quyết vấn đề này, phương pháp cập nhật các quyết định ba chiều trong hệ thống thông tin động không đầy đủ đã được đề xuất. Đầu tiên, chúng tôi xem xét mối quan hệ giữa sự thay đổi của xác suất có điều kiện đối với sự thay đổi của ba miền trong quyết định ba chiều. Sau đó, chúng tôi khảo sát sự thay đổi của xác suất có điều kiện khi các đối tượng và giá trị thuộc tính của đối tượng thay đổi. Từ sự thay đổi đó, chúng tôi đề xuất một phương pháp cập nhật các quyết định ba chiều trong hai trường hợp là khi các đối tượng thay đổi và khi các thuộc tính của một đối tượng thay đổi. Và cuối cùng, chúng tôi đưa ra một ví dụ minh họa cho phương pháp này.

Từ khoá: Tập thô; Hệ thống thông tin không đầy đủ; Quyết định ba chiều; Học tăng cường.