# A novel approach to optimize the estimated stability region via energy function for a class of nonlinear dynamical systems in technical models

Ho Thi Thu Thuy, Pham Hong Quan*

Air Defence-Air Force Academy, Kim Son, Son Tay, Ha Noi.
*Corresponding author: phamhongquan.mta@gmail.com

*Received 05 Sep 2022; Revised 26 Oct 2022; Accepted 07 Nov 2022; Published 18 Nov 2022.*
**DOI:** https://doi.org/10.54939/1859-1043.j.mst.83.2022.82-94

## ABSTRACT

*The theory of differential equations has been widely known and developed in recent years. Many researchers have drawn attention to the problem of finding the stability region of a nonlinear dynamical system in technical models, which is a complicated issue in the stability theory of dynamical systems. In this problem, how to construct an optimal energy function is considered an essential step to approximate the stability region of a locally stable equilibrium point. The main purpose of this paper is to give a novel approach to optimize the estimated stability region via energy function for nonlinear dynamical models. This ensures that the stability region estimated is optimal in the sense that this estimated region is the largest one characterized by the energy function, which lies entirely in the stability region. Furthermore, numerical experiments are also conducted to compare the difference between the proposed algorithm.*

**Keywords:** Nonlinear dynamical systems; Stability boundary; Stability region; Optimal energy function.

## 1. INTRODUCTION

It is well-known that many nonlinear physical and engineering systems are designed to be operated at an equilibrium state. In other words, they are constructed to be operated at an equilibrium point and are described by a nonlinear dynamical system. It concludes a class of nonlinear dynamical systems in technical models like electric power systems or mechanical systems in motions and elevators. The most important requirement for the successful operation of these systems is to maintain the stability of this equilibrium state. It is desired to have robust stability of the equilibrium point with respect to small perturbations. More precisely, the system state returns the equilibrium point under small perturbations. However, most physical and engineering systems are not globally stable. Therefore, the problem here is how to compute stable regions around an equilibrium point of the dynamical systems. There are a few methods that are able to estimate stability region, but most of them are based on Lyapunov functions or energy functions [1-6]. Nevertheless, one of the non-Lyapunov function approaches is level set methods and implicit dynamical surfaces [7]. In recent years, authors often focused on constructing an optimal Lyapunov function based on [8, 9]. In general, the methods often rely on building an energy function of quadratic form and approximating the stability region of nonlinear dynamical systems based on the closest unstable equilibrium method [1-3]. Depending on whether or not the nonlinear dynamical system under study has an energy function, there are two different procedures for estimating its stability regions. In this work, we will give an alternative method to construct an optimal energy function for a class of nonlinear dynamical systems that do not have a global energy function so that the stability region is approximated closest to the exact stability region. This allows us to expand the stability region in comparison with the proposed method in [1-3, 6] and [8].

In addition, one of the requirements for the closest unstable equilibrium point method in [1-3] is to assume that the nonlinear dynamical system has a global energy function, while the proposed method in this paper can deal with a class of systems without a global energy function.

The remaining of this paper is organized as follows: Section 2 presents some basic definitions of nonlinear dynamical systems in technical models. The main results of a novel approach to optimize the energy function are given in section 3, while section 4 discusses several examples to illustrate the theoretical result. The last section gives some further research and concludes the main work in this paper.

## 2. PRELIMINARIES

In this section, we first begin with reviewing some relevant concepts in the theory of the stability region for nonlinear dynamical systems. These concepts are used to depict the method more clearly in section 3. Throughout this paper, we always consider the following (autonomous) nonlinear system

$$\dot{x}(t) = f(x(t)), \tag{1}$$

where $x \in \mathbb{R}^n$ is a vector of state variables. It is natural to assume that the function $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies some sufficient conditions for the existence and uniqueness of solution to (1). The solution curve of (1) starting from $x_0$ at $t = 0$ is called the trajectory starting from $x_0$ and is denoted by $\phi(\cdot, x_0)$. The point $\hat{x} \in \mathbb{R}^n$ is said to be an equilibrium point of (1) if $f(\hat{x}) = 0$, i.e. the equilibrium point is a particular type of solution that does not change in time. Therefore, equilibrium points are degenerated trajectories that do not move, [1-3]. The set of all equilibrium points of (1) will be denoted by $E = \{x \in \mathbb{R}^n : f(x) = 0\}$. Another important type of trajectory is a closed orbit. A trajectory $\gamma$ is a closed orbit if $\gamma$ is not an equilibrium point and for any $x \in \gamma$, there exists $T > 0$ such that $\phi(T, x) = x$.

We recall that an equilibrium point $\hat{x} \in \mathbb{R}^n$ of (1) is said to be (Lyapunov) stable if for each open neighborhood $U$ of $\hat{x} \in \mathbb{R}^n$ there exists an open neighborhood $\hat{U}$ of $\hat{x} \in \mathbb{R}^n$ such that $\phi(t, x) \in U$ for all $x \in \hat{U}$ and for all $t > 0$ and otherwise, $\hat{x}$ is unstable**.** For an asymptotically stable equilibrium point $\hat{x}$, there exists $\delta > 0$ such that for any $x_0$ satisfies $\|x_0 - \hat{x}\| < \delta$ then $\phi(t, x_0) \to \hat{x}$ as $t \to +\infty$. If $\delta$ is arbitrary large then $\hat{x}$ is called a global stable equilibrium point. However, many stable equilibrium points $\hat{x}$ are not globally stable. Therefore, from now on, we only consider a local stable equilibrium point and denote it by $x^s$ in this paper. Without loss generality, we always assume that the origin is the stable equilibrium point $x^s$ that is considered. Indeed, if $x^s \neq \mathbf{0}$, we can shift it to the origin by changing the variable. The definition of the stability region of a stable equilibrium point $x^s$ is considered as follows.

**Definition 1.** The stability region of a stable equilibrium point $x^s$ for the nonlinear autonomous dynamical system (1), denoted $A(x^s)$, is defined as follows

$$A\left(x^s\right) := \left\{x \in \mathbb{R}^n : \lim_{t\to+\infty} \phi\left(t, x\right) \to x^s\right\}. \quad (2)$$

Clearly, the stability region can be expressed as $A\left(x^s\right) = \left\{x \in \mathbb{R}^n : w\left(x\right) = x^s\right\},$ where $w(x)$ is the $w$-limit set of $x$. The stability boundary of $x^s$ is the boundary of the stability region $A\left(x^s\right)$ and is denoted by $\partial A\left(x^s\right)$. In other words, every trajectory starting in stability region lies entirely in it. Since the boundary of an invariant set is also invariant and the boundary of an open set is a closed set, we can conclude that the stability boundary $\partial A\left(x^s\right)$ is a closed invariant set of dimension less than $n$. If $A\left(x^s\right)$ is not dense in $\mathbb{R}^n$ then $\partial A\left(x^s\right)$ is of dimension $n-1$.

Before giving the main results of energy functions, we recall the definition of an energy function as follows.

**Definition 2.** We said that a $C^r$-function $V : \mathbb{R}^n \to \mathbb{R},\ r \geq 1,$ is an *energy function* for the system $\dot{x} = f\left(x\left(t\right)\right)$ if it satisfies three following conditions. That is,

(E1) The derivative of the function $V\left(x\right)$ along any system trajectory $x\left(t\right)$ is non-positive, i.e.

$$\dot{V}\left(x\left(t\right)\right) \leq 0.$$

(E2) If $x\left(t\right)$ is a nontrivial trajectory ( $x\left(t\right)$ is not an equilibrium point) then along the nontrivial trajectory $x\left(t\right),$ the set $\left\{t \in \mathbb{R} : \dot{V}\left(x\left(t\right)\right) = 0\right\}$ has measured zero in $\mathbb{R}$.

(E3) A trajectory $x\left(t\right)$ has a bounded value of $V\left(x\left(t\right)\right)$ for $t \in \mathbb{R}^+$ implies that the trajectory $x\left(t\right)$ is also bounded for $t \in \mathbb{R}^+$.

**Remark 3.** In general, the concepts of energy function and Lyapunov function of a nonlinear dynamical system are not identical. In some cases, a function $V\left(x\right)$ may be a Lyapunov function but not an energy function. Moreover, the nonlinear dynamic system (1) can have many energy functions.

**Proposition 4.** The sum of the energy functions for (1) is also an energy function.

*Proof.* Without loss generality, we will prove the proposition for the case of two energy functions. Assume that $V_1\left(.\right)$ and $V_2\left(.\right)$ are two energy functions of the nonlinear dynamical system (1). Consider $V := V_1\left(.\right) + V_2\left(.\right)$ is also an energy function of (1). Indeed, since $\dot{V}_1\left(x\left(t\right)\right) \leq 0$ and $\dot{V}_2\left(x\left(t\right)\right) \leq 0,$ we have

$$\dot{V} = a\dot{V}_1\left(.\right) + b\dot{V}_2\left(.\right) \leq 0.$$

Hence, (E1) is satisfied. Since $V_1$ and $V_2$ are two energy functions, it is easy to deduce that (E2) and (E3) hold. Therefore, $V$ is an energy function of the nonlinear dynamical system (1).

□

For nonlinear dynamical systems that have a global energy function, the estimated stability region relies on the value of the closest unstable equilibrium point on the stability boundary. For our problem, we suppose that the given nonlinear system does not have a global energy function. Consequently, we need to establish a local energy function for (1) before approximating the stability region. Now, we will show that there exists an optimal energy function $V_m$ that can be used to estimate $A\left(x^s\right)$ exactly. Then, we also derive a partial differential equation characterizing $V_m$, and propose an iterative method for solving the partial differential equation. This iterative method leads to a new recursive technique for approximating the stability region of $x^s$ for (1).

We now take an energy function as the following form into consideration

$$V\left(x\right)=\frac{N\left(x\right)}{D\left(x\right)}. \tag{3}$$

Note that, the above form of $V(x)$ is also considered for a Lyapunov function in [6] and [11]. In particular, this form use radial basis functions. In fact, the algorithm works well even in the case of the unbounded stability region $A(x^s)$ and in practical computation, the first few iterations usually show whether the stability region is bounded or unbounded. After that, the stability boundary can be found by solving the equation

$$D\left(x\right)=0. \tag{4}$$

The following results of this section is to show the existence of an optimal energy function $V_m$ that can be used to estimate the stability region $A\left(x^s\right)$. They are reformulated from [6] and [11] with other versions and we will not give the detail proof here.

**Theorem 5.** Suppose we can find a set $A\subset\mathbb{R}^n$ containing the origin in its interior, a continuous function $V:A\to\mathbb{R}$ and a positive definite function $\varphi$ such that

(i) $V\left(\mathbf{0}\right)$ is the lowest value of $V\left(x\right)$ in $A$.

(ii) The function

$$\dot{V}\left(x_0\right)=\lim_{t\to 0^+}\frac{\left[V\left(x\left(x_0,t\right)\right)-V\left(x_0\right)\right]}{t}$$

is well-defined at all $x\in A$, and satisfies $\dot{V}\left(x\right)=-\varphi\left(x\right)$ for all $x\in A$.

(iii) $V\left(x\right)\to\infty$ as $x\to\partial A$ or as $\left|x\right|\to\infty$.

Then $A=A(x^s)$.

**Corollary 6.** Suppose we can find a set $A\subset\mathbb{R}^n$ containing the origin in its interior, a continuously differentiable function $V:A\to\mathbb{R}$, and a positive definite function $\varphi$ such that

(i) $V\left(\mathbf{0}\right)$ is the lowest value of $V\left(x\right)$ in $A$.

(ii) $\left\langle\nabla V(x), f(x)\right\rangle=-\varphi\left(x\right)$ for all $x\in A$.

(iii) $V(x) \to \infty$ as $x \to \partial A$ or as $|x| \to \infty$.

Then $A = A(x^s)$.

In our assumptions, $f$ satisfies some sufficient conditions for the existence and uniqueness of solution to (1). One of these conditions is that $f$ is continuously differentiable in some neighborhood of the origin. Thus, the conditions on $V$ imposed in Theorem 5 and Corollary 6 are reasonable.

**Theorem 7** ([6]). Suppose $f$ is Lipschitz continuous on a set that contains $A(x^s)$. Then, in order for an open set $A$ containing the origin to be the stability region for (1), the necessary and sufficient condition is that there exist a continuous function $V: A \to \mathbb{R}$ and a positive definite function $\varphi$ such that the conditions (i)-(iii) of Theorem 5 are satisfied.

The following result is a direct corollary of above theorem.

**Corollary 8** ([6]). Suppose $f$ is Lipschitz continuous on $A(x^s)$. Then, in order for an open set $A$ containing $\mathbf{0}$ to be the stability region for (1), it is necessary and sufficient that there exist a continuous function $V: A \to \mathbb{R}$ and a positive definite function $\varphi$ such that

(i) $W(\mathbf{0}) = 0$, $W(x) < 0$ for all $x \in A \setminus \{\mathbf{0}\}$;

(ii) $W(x) \to -1$ as $x \to \partial A$ and/or $|x| \to \infty$;

(iii) $\dot{W}(x)$ is well-defined at all $x \in A$ and satisfies

$$\dot{W}(x) = \frac{1 + W(x)}{x}. \tag{5}$$

In some research, some authors often suppose $V(x^s) = 0$ or $V(\mathbf{0}) = 0$. However, we did not impose this condition in our problem because the definition of an energy function, which is constructed in this paper, accepts negative values. Before giving computation of an optimal energy function, we have next results which are motivated by the fact that it is usually quite easy to find functions $V$ and $\varphi$ such that (5) is satisfied in some neighborhood of $\mathbf{0}$. Lemma 9 shows that such a function can be extended to all of $A(x^s)$.

**Lemma 9** ([10]). Suppose $V$ is a continuous function on some ball $B$ such that $V(\mathbf{0})$ is the lowest value of $V(x)$ in $A(x^s)$ and $\dot{V}$ is negative definite. Then, $V$ is positive definite.

The importance of Lemma 9 is in showing that if we can find a function $V(x)$ and a positive definite function $\varphi(x)$ such that $V(\mathbf{0}) = 0$ and

$$\langle \nabla V(x), f(x) \rangle = -\varphi(x),$$

Then $V$ is guaranteed to be positive definite.

Now, we present the definition of an optimal energy function.

**Definition 10 (**[6]). A function $V_m : \mathbb{R}^n \to \mathbb{R}$ is called an optimal energy function for the system (1) if the following conditions are satisfied.

(i) $V_m(\mathbf{0})$ is the lowest value of $V(x)$ for all $x \in A(x^s) \setminus \{0\}$.

(ii) $V_m(x) < \infty$ if an only if $x \in A(x^s)$.

(iii) $V_m(x) \to \infty$ as $x \to \partial A(x^s)$ and/or $|x| \to \infty$.

(iv) $\dot{V}_m$ is a well-defined and negative definite function over $A(x^s)$.

Lemma 11 below shows that any energy function for the system (1) can be extended to an optimal energy function.

**Lemma 11** ([10])**.** Suppose $f$ is Lipschitz on $A(x^s)$, and suppose $V$ is continuous function on some closed ball $\bar{B}$ such that $V(\mathbf{0})$ is the lowest value of $V(x)$ and $\dot{V}$ is negative definite on $\bar{B}$. Then there exists an optimal energy function $V_m$ that agrees with $V$ on $\bar{B}$.

## 3. CALCULATION OF ENERGY FUNCTION

Based on the theoretical background in Section 2, we now propose a method to look for an optimal energy function $V$ of the nonlinear dynamical system (1) so that $V(\mathbf{0})$ is the lowest value of $V(x)$ and

$$\dot{V}(x) = -\varphi(x),$$

in some neighborhood of the origin. Then the stability boundary is defined by $V(x) \to \infty$. It also means that we can try to look for a function $W$ and a positive definite function $\varphi$ such that $W(0) = 0$ and

$$\dot{W}(x) = \frac{1 + W(x)}{\varphi(x)}. \tag{6}$$

In this case, the stability boundary is defined by $W(x) = -1$. Now, we will show that (6) and equation $\dot{V}(x) = -\varphi(x)$ are equivalent. Indeed, if $V(.)$ is solved from $\dot{V}(x) = -\varphi(x)$, it follows that $W = e^{-V} - 1$. Conversely, if $W(.)$ is solved from (6), we have $V = \ln(1 + W)$.

In the following, we present a systematic procedure for solving $\dot{V}(x) = -\varphi(x)$, where $f$ is an analytic function. Recall that we can rewrite $f$ as follows

$$f(x) = \sum_{i=1}^{\infty} F_i(x), \tag{7}$$

where $F_i(.)$ is a homogeneous function of degree $i$. For instance,

$$F_1(x) = Ax,\ A \in \mathbb{R}^{n \times n}.$$

In section 4, we will give a detail way to determine matrix $A$. A potential energy function candidate is suggested by condition (iii) of Definition 10. The candidate must in effect “blow up” near the stability boundary $\partial A\left(x^s\right)$, [10-11]. This is different to others which have the property that they are defined on $\mathbb{R}^n$. Therefore, in order to obtain the desired effect, we write the function $V(x)$ as the form (3).

If $V(x)\to\infty$ as $x\in\partial A(x^s)$, it follows that $x\in\partial A(x^s)$ when $D(x)=0$. Then, the stability boundary $\partial A\left(x^s\right)$ is defined by solving equation $D(x)=0$. In order to do this task, the function $V(x)$ is rewritten as the following form

$$V(x)=\frac{\sum_{i=2}^{\infty}R_i}{1+\sum_{i=1}^{\infty}Q_i}, \quad (8)$$

where $R_i$, $Q_i$ are homogeneous functions of degree $i$. For the purpose $V(x)=-\varphi(x)$, we have

$$\dot{V}(x)=\langle\nabla V(x), f(x)\rangle=-\varphi(x)=-x^TQx,$$

where $Q$ is a symmetric, positive define matrix, [2] and [7], we must have

$$\left[\left(1+\sum_{i=1}^{\infty}Q_i\right)\sum_{i=2}^{\infty}\nabla R_i-\left(\sum_{i=1}^{\infty}\nabla Q_i\right)\sum_{i=2}^{\infty}R_i\right]\sum_{i=1}^{\infty}F_i=-x^TQx\left(1+\sum_{i=1}^{\infty}Q_i\right)^2.$$

Finally, we have

$$\sum_{i=2}^{\infty}\sum_{k=1}^{\infty}\nabla R_iF_k+\sum_{i=1}^{\infty}\sum_{j=2}^{\infty}\sum_{k=1}^{\infty}Q_i\nabla R_jF_k=\sum_{i=1}^{\infty}\sum_{j=2}^{\infty}\sum_{k=1}^{\infty}\nabla Q_iR_jF_k$$
$$=-x^TQx\left(1+2\sum_{i=1}^{\infty}Q_i+\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}Q_iQ_j\right).$$

By comparing the coefficients of the degrees of the left and right hand sides of the above equation, for degree 2, we obtain

$$\nabla R_2{}^TF_1=-x^TQx.$$

In general, due to the fact that the recursive relation for degree two is quite trivial, we should find that the recursive relation for degree greater than or equal to 3 is

$$\sum_{i=1}^{k}\nabla R_iF_{k+1-i}+\sum_{i=1}^{k-2}\sum_{j=2}^{k-1}\left(Q_i\nabla R_j-\nabla Q_iR_j\right)F_{k+1-i-j}$$
$$=-x^TQx\left(2Q_{k-2}+\sum_{i=1}^{k-3}Q_iQ_{k-2-i}\right). \quad (9)$$

In which the recursive relations are equivalent to solving the linear equation systems

$$A_ny=b_n,$$

where $A_n$ is an $n \times n$ matrix, the recursive relation in (9) is equivalent to a singular linear equation system with $m < n.$ Therefore, the recursive relation (9) yields several degrees of freedom in choosing $R_i,\ Q_i$ for (8). This fact is exploited in the following procedure.

Since $f(x)$ is the sum of homogenous functions $F_i(x),$ we consider another form of the nonlinear system of equations (1) as follows

$$\dot{x} = f(x) = \sum_{i=1}^{\infty} F_i(x).$$

We choose homogeneous functions $R_n$ and $Q_{n-2},\ n \geq 3,$ such that the coefficients of $R_n$ and $Q_n$ are determined by solving the following constrained minimization problem

$$\begin{aligned} &\min e_n(y) \\ &\text{s.t}\ A_n y = b_n, \end{aligned} \tag{10}$$

where $A_n y = b_n$ is the equivalent form of (9) and $e_n(y)$ is the norm of the coefficients of the residual terms of $\dot{V}_n.$

After that, one can choose the largest positive value $d^*$ such that the level set

$$V_n = \frac{R_2 + R_3 + \cdots R_n}{1 + Q_1 + \cdots Q_{n-2}} = d^*, \tag{11}$$

is contained in the region given by $\{x : \dot{V}_n(x) \leq 0\}.$ Then, the set

$$SR(d^*) = \{x : V_n(x) < d^*\},$$

is contained in the stability region $A(x^s).$

If $e_n(y^*) = 0$ for some $y^*$ that satisfies $A_n y = b_n,$ then

$$\dot{V}_n = -x^T Qx,$$

where $Q$ is a symmetric, positive define matrix. According to Theorem 1-(iii), we can conclude that the stability region $A(x^s)$ is determined by

$$A(x^s) = \left\{x : \sum_{i=1}^{n-2} Q_i > -1\right\}. \tag{12}$$

The above procedure suggests the following algorithm for estimating of the stability region.

**Algorithm**

*Step 1*. Since $F_1 = Ax,$ find $P$ by solving

$$A^T P + PA = -Q,$$

then,

$$V_2(x) = R_2 = x^T Px.$$

In general, we often choose $Q = I_n,$ where $I_n$ is the identity matrix.

*Step 2.* Find the linear system representation for $n(n \geq 3)$, i.e. find

$$A_n y = b_n.$$

*Step 3.* Define $e_n(y)$ as in (10) and solve the constrained minimization problem (10). Assume that $y^*$ is the solution.

*Step 4.* Use $y^*$ to determine coefficients of $R_n$ and $Q_{n-2}$. If $e_n(y)$ is sufficiently small, go to *Step 5*; else, we increase $n$, Repeat *Step 2* and *Step 3*.

*Step 5.* If $e_n(y^*)$ is small enough go to Step 6; else, solve the constrained optimization problem given by (10). Let $d^*$ be the optimal value of the objective function. Then, we compute

$$SR(d^*) = \{x : V_n(x) < d^*\},$$

is an approximation to $A(x^s)$ such that $SR(d^*) \subset A(x^s)$.

*Step 6.* The set $SR := \left\{ x : \sum_{i=1}^{n-2} Q_i > -1 \right\}$ is an approximation to $A(x^s)$.

## 4. NUMERICAL EXPERIMENT AND DISCUSSION

In this section, we will present two examples for illustrating the above results. The first example is considered in [1-3] and [10-14] while the second is introduced [6] and [10]. In addition, we analyze our results in comparison with the numerical results in [1-3]. The local energy functions in [1-3] are constructed in the simplest quadratic form then the level energy surface is used to approximate the stability region.

**Example 1.** Consider the following system

$$\dot{x} = -y$$
$$\dot{y} = x - (1 - x^2)y.$$

This is the well-known van der Pol equation that describes triode oscillations in electrical circuits; here $(0,0)$ is a stable equilibrium point of this equation whose stability region is to be estimated.

Firstly, from direct computations, we have

$$R_2 = \frac{3}{2}x^2 - xy + y^2,$$
$$R_3 = 0,$$
$$R_4 = -0.3186x^4 + 0.7124x^3 y - 0.1459x^2 y^2 + 0.1409xy^3 - 0.03769y^4,$$
$$Q_1 = 0,$$
$$Q_2 = -0.2362x^2 + 0.31747xy - 0.1091y^2.$$

Using $e_4 = 0.12295$ and

$$V_4(x)=\frac{R_2+R_3+R_4}{1+Q_1+Q_2}$$
$$=\frac{0.593x^2-0.364xy++0.437y^2-0.1253x^4+0.2885x^3y-0.0537x^2y^2+0.0581xy^3-0.0196y^4}{1-0.0001x+0.001y-0.2685x^2+0.3217xy-0.1163y^2}.$$

We obtain the critical value is $d^*=5.4413$. Thus, $SR(d^*)=\{x:V_4(x)<5.4413\}$ is a subset of the stability region of the stable equilibrium point $x^s=(0,0)$ for the given system. Figure 1 illustrates the phase portrait and compares the estimated stability region by our method (the dashed region) and the stability boundary by the proposed method in [1-3] (the green line) and the exact stability boundary (the red line). Looking at figure 1 in more detail, the stability region estimated via an optimal energy function is approximated closest to the exact stability region.

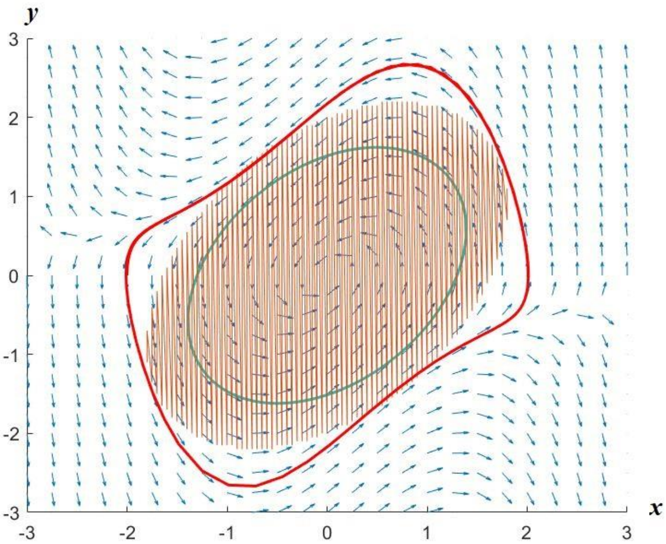


***Figure 1.*** *The exact stability region and the estimated stability region.*

**Example 2**. Consider a planar system as follows

$$\dot{x}=-x\left((x-1)(x-3)+\frac{1}{2}y\right)$$
$$\dot{y}=y(-2.1+x).$$

This example illustrates that the proposed algorithm can be applied to systems with multiple equilibrium also. The system in this example is called the Lotka-Volterra system also known as the predator-prey equations in biological population models It is easy to see that the above system has two stable equilibrium points: $x^s=(0,0)$ and $x^s=(2.1,1.98)$. Since the origin is trivial, we now attempt to estimate the stability region of the stable equilibrium point $x^s=(2.1,1.98)$. In order to apply the propose method, we need shift this point to the origin by changing variable $u=x-2.1$, $v=y-1.98$ and the planar system has the following form

$$\dot{u} = -0.42u - 1.05v - 2.3u^2 - \frac{1}{2}uv - u^3$$
$$\dot{v} = 1.98u + uv$$

We have $Q_1 = 0.6416u - 0.7563v,\ Q_2 = -0.5346u^2 - 0.7059uv + 0.02576v^2$ and

$$R_2 = 3.435u^2 + 0.9524uv + 1.923v^2$$
$$R_3 = -0.8839u^3 + 4.029u^2v + 0.5133uv^2 + 0.1054v^3$$
$$R_4 = 0.1622u^4 - 0.6530u^3v + 1.354v^3 - 0.3321uv^3 + 0.2719v^4.$$

Using $e_4 = 114.62$ and

$$V_4(x) = \frac{R_2 + R^3 + R^4}{1 + Q_1 + Q_2}$$
$$= \frac{0.1622u^4 - 0.8839u^3 + 3.435u^2 + 4.029u^2v + 0.9524uv}{1 - 0.5346u^2 + 0.6416u - 0.7059uv - 0.7563v + 0.02576v^2}$$
$$+ \frac{0.5133uv^2 - 0.3321uv^3 + 1.923v^2 + 0.24008v^3 + 0.2719v^4}{1 - 0.5346u^2 + 0.6416u - 0.7059uv - 0.7563v + 0.02576v^2}.$$

The critical value is determined that $d^* = 1.1$. Therefore, $SR(d^*) = \{x : V_4(x) < 1.1\}$ is a subset of the stability region of the planar system. In order to obtain the stability region of $x^s = (2.1, 1.98)$ in the original system, we need to shift $SR(d^*)$ along $(2.1, 1.98)$. Figure 2 illustrates the estimated stability region by our method (the dashed region) and the estimated stability boundary by the proposed method in [1-3] (the green line).

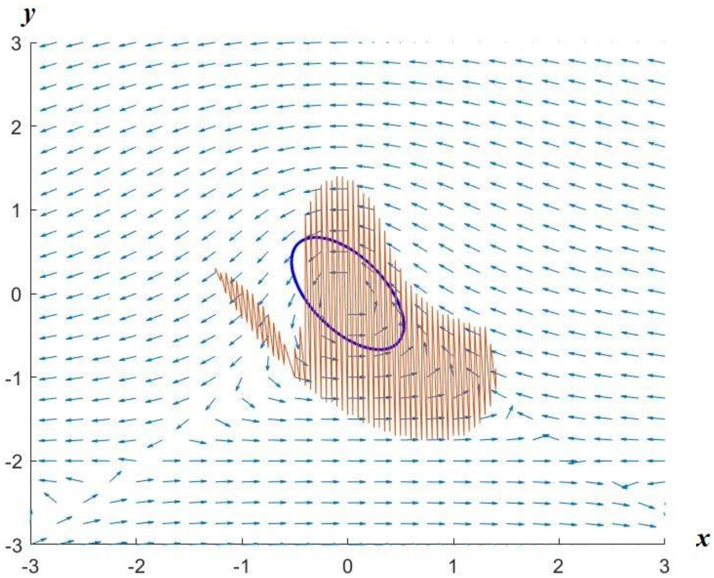


***Figure 2.*** *The exact stability region and the estimated stability region.*

Clearly, the stability region estimated via an optimal energy function in our method contains almost that estimated by the previous method in [1-3]. Therefore, it is approximated as closest to the exact stability region in comparison with previous methods.

## 5. CONCLUSIONS

In this paper, we offered a novel method to construct an optimal energy function for approximating the stability region. Another point to consider in this work is that we have presented remarkable properties for predicting the stability boundary and the stability region of a stable equilibrium point of nonlinear dynamical systems. Furthermore, numerical experiments are also carried out to illustrate the efficiency of the new approach in comparison with different methods. Our results can be extended by combining this proposed method and the method in [2] in order to estimate the stability region of nonlinear dynamical systems. We also expect that our approach is compatible with nonlinear dynamical systems that have a global energy function. On the other hand, we can also construct the optimal energy function with higher order; however, the computation cost is expensive. Therefore, we recommend that the polynomial energy functions of order 4 are enough.

## REFERENCES

[1]. H. D. Chiang, *"Direct methods for stability analysis of electric power systems: theoretical foundation, BCU methodologies, and applications"*, John Wiley & Sons, Hoboken, (2011).

[2]. H. D. Chiang, J. S. Thorp, *"Stability regions of nonlinear dynamical systems: A constructive methodology"*, IEEE Transactions on Automatic Control, Vol. **34**, No. 12, pp. 1229-1241, (1989).

[3]. H. D. Chiang and L. F. C Alberto, *"Stability regions of nonlinear dynamical systems: theory, estimation, and applications"*, Cambridge University Press, Cambridge, (2015).

[4]. M. Sassano and A. Astolfi, Dynamic lyapunov functions, *Automatica*, Vol. **49**, No. 4, pp. 1058-1067, (2013).

[5]. E. D. Ferreira and B. H. Krogh, *"Using neural networks to estimate regions of stability"*, Proceedings of the 1997 American Control Conference, Vol. **3**, pp. 1989-1993, (1997).

[6]. A. Vannelli and M. Vidyasagar, *"Maximal Lyapunov functions and domains of attraction for autonomous nonlinear systems"*, Automatica, Vol. **21**, No. 1, pp. 69-89, (1985).

[7]. T. J. Koo, H. Su, *"A computational approach for estimating stability regions"*, 2006 IEEE International Symposium on Intelligent Control, pp. 62-68, (2006).

[8]. V. I. Zubov, *"Mathematical methods for the study of automatic control systems"*, PergmSon Press, (1963).

[9]. P. Giesl and S. Hafstein, *"Review on computational methods for Lyapunov functions"*, Discrete and Continuous Dynamical Systems-Series B, Vol. **20**, No. 8, pp. 2291-2331, (2015).

[10]. L. G. Matallana, A. M. Blanco and J. A. Bandoni, *"Estimation of domains of attraction: A global optimization approach"*, Mathematical and Computer Modeling, Vol. **52**, No. 3-4, pp. 574-585, (2010).

[11]. P. Giesl, *"Approximation of domains of attraction and Lyapunov functions using radial basis functions"*, IFAC Proceedings Volumes, Vol. **37**, No. 13, pp. 697-702, (2004).

[12]. W. Tan and A. Packard, *"Stability region analysis using polynomial and composite polynomial Lyapunov functions and sum of squares programming"*, IEEE Transactions on Automatic Control, Vol. **53**, No. 2, pp. 565-571, (2008).

[13]. G. Yuan and Y. Li, *"Estimation of the regions of attraction for autonomous nonlinear systems"*, Transactions of the Institute of Measurement and Control, Vol. **41**, No. 1, pp. 97-106, (2019).

[14]. M. P. Rezaiee and B. Moghaddasie, *"Determination of stability domains for nonlinear dynamical systems using the weighted residuals method"*, Civil Engineering Infrastructures Journal, Vol. **46**, No. 1, pp. 27-50, (2013).

## TÓM TẮT

### Một cách tiếp cận mới để tối ưu miền ổn định ước lượng theo hàm năng lượng đối với một lớp các hệ động lực phi tuyến trong các mô hình kỹ thuật

*Lý thuyết về phương trình vi phân được biết đến rộng rãi và phát triển trong nhiều năm gần đây. Nhiều nhà nghiên cứu đã dành sự quan tâm đến bài toán tìm miền ổn định của hệ động lực phi tuyến trong các mô hình kỹ thuật, đây là một vấn đề phức tạp trong lý thuyết ổn định của hệ động lực. Trong bài toán này, làm sao để xây dựng một hàm năng lượng tối ưu được xem như là bước cốt yếu để xấp xỉ miền ổn định của một điểm cân bằng địa phương. Mục đích của bài báo này là đưa ra một cách tiếp cận để tối ưu miền ổn định ước lượng theo hàm năng lượng đối với các mô hình hệ động lực. Điều này đảm bảo miền ổn định ước lượng theo hàm năng lượng là tối ưu theo nghĩa lớn nhất và nằm hoàn toàn trong miền ổn định chính xác. Hơn nữa, các thử nghiệm số cũng được tiến hành để so sánh sự khác biệt của thuật toán đề xuất.*

**Từ khóa:** Hệ động lực phi tuyến; Biên ổn định; Miền ổn định; Hàm năng lượng tối ưu.