Asymptotic stability of dynamical systems with Barbalat's lemma and Lyapunov function

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ABSTRACT

The article explains Barbalat's lemma, combining the application of Barbalat's lemma, the Lyapunov function, and the theorem Lagrange to ensure mathematical certainty in analyzing the asymptotic stability of a non-autonomous control system. Research results are illustrated and simulated with visual examples of uncontrolled and controlled dynamical systems.

Keywords: Lemma Barbalat; Lyapunov function; Theorem Lagrange; Asymptotic stability; Non-autonomous system.

1. INTRODUCTION

The birth of Lyapunov theory has been making an important contribution to ensuring mathematical stability in analyzing the asymptotic stability of a dynamical system. To apply this theory, usually, the designer needs to choose a certain Lyapunov function $V(t)$. A true Lyapunov function $V(t)$ requires itself to be positive ($V > 0$), and its derivative must be negative ($\dot{V} < 0$) [1]. Thus, when only ensuring the derivative of $V(t)$ is negative semi-deterministic ($\dot{V} \leq 0$), it is immediately concluded the asymptotically stable system does not have enough mathematical basis.

For autonomous systems, LaSalle's invariant set principles are powerful tools for studying stability, as they allow conclusions about asymptotic stability even if the term $\dot{V}$ is only negative semi-deterministic [3]. However, we cannot apply the invariant set principles to non-autonomous systems. Instead, we need to find a Lyapunov function with negative deterministic differentiation. So main difficulty in analyzing the stability of dynamic systems is to choose an appropriate Lyapunov candidate function and calculate its differentiation. An important and straightforward result that partially overcomes this situation is Barbalat's lemma [2-4]. This lemma became popular due to its applicability in the analysis of asymptotic stability of time-varying nonlinear systems [5-7]. Barbalat's lemma is a purely mathematical statement concerning the asymptotic properties of functions and their derivatives. When used appropriately for dynamical systems, especially non-autonomous systems, it can lead to satisfactory solutions to many asymptotic stability problems.

This paper interprets Barbalat's lemma with specific analysis by precise mathematical calculation, intending to contribute to clarifying an analytical tool and synthesizing an asymptotic stable control system based on Lyapunov stability theory. Research results are illustrated and simulated by some visual examples.

2. BARBALAT'S LEMMA AND THE BOUNDED PROPERTY OF A LYAPUNOV-FORM FUNCTION

2.1. Barbalat's lemma

Barbalat's lemma states that if a function $f(t)$ is time-dependent, bounded, differentiable, and has a uniformly continuous differentiation, then its differentiation will converge to zero [2]. Of the several ways proofs have been reinterpreted, proof by contradiction is the most common.

Statement:

Suppose $f(t) \in R(a,\infty)$ *and* $\lim_{t\to\infty} f(t) = \alpha;\ \alpha < \infty$. *If* $\dot{f}(t)$ *is uniformly continuous, then* $\lim_{t\to\infty} \dot{f}(t) = 0$.

Proof:

Using counter-hypothesis: Suppose $\lim_{t\to\infty} \dot{f}(t) \neq 0$.

Then there exists a number $\epsilon > 0$ and a monotonous increasing sequence $\{t_n\}$ satisfies:

$$\begin{cases} t_n \to \infty : \ n \to \infty \\ |\dot{f}(t_n)| \geq \epsilon \ \forall n \in N \end{cases}$$

Because $\dot{f}(t)$ is a uniformly continuous function, then for any number $\epsilon/2$ will exists a number $\delta > 0$ such that with $\forall n \in N$:

$$|t - t_n| < \delta \ \Rightarrow |\dot{f}(t) - \dot{f}(t_n)| < \frac{\epsilon}{2} \ \Rightarrow -|\dot{f}(t) - \dot{f}(t_n)| > -\frac{\epsilon}{2}$$

So, if $t \in [t_n, t_n + \delta]$ then:

$$|\dot{f}(t)| = |\dot{f}(t_n) - [\dot{f}(t_n) - \dot{f}(t)]| \geq |\dot{f}(t_n)| - |[\dot{f}(t_n) - \dot{f}(t)]| \geq \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2}$$

$$\Rightarrow |\dot{f}(t)| \geq \frac{\epsilon}{2} \tag{1}$$

Next, since $f(t) \in R(a,\infty)$ so:

$$\left|\int_a^{t_n+\delta} \dot{f}(t)dt - \int_a^{t_n} \dot{f}(t)dt\right| = \left|\int_{t_n}^{t_n+\delta} \dot{f}(t)dt\right| \geq \int_{t_n}^{t_n+\delta} |\dot{f}(t)|dt \tag{2}$$

Combine (1) and (2):

$$\left|\int_a^{t_n+\delta} \dot{f}(t)dt - \int_a^{t_n} \dot{f}(t)dt\right| \geq \int_{t_n}^{t_n+\delta} \frac{\epsilon}{2} dt = \frac{\epsilon\delta}{2} \tag{3}$$

However, from the assumption of the theorem, one has:

$$\begin{aligned} \lim_{n\to\infty} \left|\int_a^{t_n+\delta} \dot{f}(t)dt - \int_a^{t_n} \dot{f}(t)dt\right| &= \lim_{n\to\infty} |f(t_n + \delta) - f(t_n)| \\ &= \lim_{n\to\infty} |f(t_n + \delta)| - \lim_{n\to\infty} |f(t_n)| \\ &= |\alpha| - |\alpha| = 0 \end{aligned} \tag{4}$$

Result (3) from the counter-hypothesis assumption contradicts the result (4) derived from the theorem's hypothesis. So, the counter-hypothesis belief does not exist. Therefore $\lim_{t\to\infty} \dot{f}(t) = 0$.

The lemma has been proved. ∎

Remark: $\lim_{t\to\infty} \dot{f}(t) = 0$ means that $\lim_{t\to\infty} \dot{f}(t)$ exists and equals 0. According to the limit definition, when $\lim_{t\to\infty} \dot{f}(t) = 0$, then for any number $\epsilon > 0$ (arbitrarily small), there is always a number p such that $\forall n > p$ to have $\left|\dot{f}(t_n) - 0\right| < \epsilon$ or $\left|\dot{f}(t_n)\right| < \epsilon$, means that $\dot{f}(t)$ converges.

2.2. Rolle's theorem

Rolle's theorem says that if a function is continuous over an interval, differentiable over that range, and whose two endpoints are equal, there exists a point (in the considered range) where the function's derivative is zero [8].

Statement:

If f(x) is a continuous function over an interval [a,b], differentiable over the range (a, b) and $f(a) = f(b)$, then there exists a number $c \in (a, b)$ such that $\dot{f}(c) = 0$.

Proof:

Because $f(x)$ is continuous over $[a, b]$ so, according to the Weierstrass' theorem on the extremes existence of a continuous function, then $f(x)$ gets the maximum value M and the minimum value m over $[a, b]$.

If $M = m$ then $f(x)$ is the constant function over $[a, b]$, so with $\forall c \in (a, b)$, there's always $\dot{f}(c) = 0$.

When $M > m$, since $f(a) = f(b)$ so there exists $c \in (a, b)$ such that $f(c) = M$ or $f(c) = m$. Meaning the coordinate $\big(c, f(c)\big)$ is the local extreme of $f(x)$ over (a, b). Because $f(x)$ is differentiable over (a, b) so according to Fermat's theorem on local extremes, one has $\dot{f}(c) = 0$.

The theorem has been proved. ∎

Remark: Roll's theorem is the basis to prove Lagrange's theorem in section 2.3.

2.3. Lagrange's theorem

Lagrange's theorem, also known as the mean value theorem [8], is stated as follows.

Statement:

Give the function $f(x)$: $[a, b] \to R$, which is continuous over $[a, b]$ and differentiable over (a, b). There exists a real number c such that:

$$\frac{f(b) - f(a)}{b - a} = \dot{f}(c) \tag{5}$$

Proof:

Consider the function:

$$F(x) = f(x) - \frac{f(b) - f(a)}{b - a} x \tag{6}$$

We find that $F(x)$ is continuous over $[a, b]$, differentiable over (a, b) and $F(a) = F(b)$. According to Roll's theorem, there exists $c \in (a, b)$ such that $\dot{F}(c) = 0$.

On the other hand:

$$\begin{gathered} \dot{F}(x) = \dot{f}(x) - \frac{f(b) - f(a)}{b - a} \\ \dot{F}(c) = 0 \Rightarrow \dot{f}(c) = \frac{f(b) - f(a)}{b - a} \end{gathered} \tag{7}$$

The theorem has been proved. ∎

Remarks:

a) From formula (5) of Lagrange's theorem, finding that:

- If $\dot{f}(x) > 0$ with $\forall x \in (a,b)$ then $f(x)$ is an increasing monotonical function over (a,b).

- If $\dot{f}(x) < 0$ with $\forall x \in (a,b)$ then $f(x)$ is a decreasing monotonical function over (a,b).

- If $\dot{f}(x) = 0$ with $\forall x \in (a,b)$ then $f(x)$ is a constant function over (a,b).

b) It is possible to apply this Lagrange's theorem to confirm the bounded property of a form of Lyapunov functions. This content presents in section 2.4 below.

2.4. The bounded property of a Lyapunov-form function

This section confirms the boundedness of a sliding function S together with a Lyapunov-form function V of a sliding function S, to determine the bounded property for the second derivative of the Lyapunov function, creating a mathematical basis for the application of Barbalat's lemma to analyze the asymptotic stability of a control system.

Suppose we choose a Lyapunov function $V(t)$ of the form (8), where $S = 0$ is an autonomous system that includes all the time-dependent state variables of the considered control system.

$$V = S^2 \tag{8}$$

Suppose we synthesize the controller u which ensures the derivative of S of the form (9):

$$\dot{S} = -\varepsilon \text{sign}(S) - kS;\ \varepsilon, k > 0 \tag{9}$$

Under the two assumptions above, we state a novel Lemma below:

Lemma (on the boundedness of the Lyapunov function and sliding function):

If the controlled system has the continuous sliding function S with the sliding surface approaching law (9), then the Lyapunov function (8) and the function S are always bounded.

Proof:

From (8), derive $V(t)$ to time:

$$\dot{V} = 2S\dot{S} \tag{10}$$

Substitute (9) into (10):

$$\dot{V} = -2\varepsilon|S| - 2k|S|^2 \leq 0\ \forall S \tag{11}$$

From (11), according to Lagrange's theorem, $V(t)$ is a decreasing function (monotonically decreasing) with $\forall S \neq 0$, which means that when comparing $V(t)$ at the initial time $t = 0$ and later time $t > 0$, there's always $V(t) \leq V(0)$.

When $V(t) \leq V(0)$ means that Lyapunov $V(t)$ is always bounded.

Also, from (8) deduces:

$$\begin{gathered} |S| = \sqrt{V} \leq \sqrt{V(0)} \\ \Rightarrow -\sqrt{V(0)} \leq S \leq +\sqrt{V(0)} \end{gathered} \tag{12}$$

So $V(t)$ and $S(t)$ are bounded.

The lemma has been proved. ■

3. APPLICATION OF BARBALAT'S LEMMA TO ANALYZE ASYMPTOTIC STABILITY OF DYNAMICAL SYSTEMS

3.1. Analysis of asymptotic stability of an uncontrolled system

Consider system (13) with sate variables $e(t)$ and $g(t)$ [3]:

$$\begin{cases} \dot{e}(t) = -e(t) + g(t)\omega(t) \\ \dot{g}(t) = -e(t)\omega(t) \end{cases} \tag{13}$$

System (13) is a non-autonomous system because the input $\omega(t)$ is a time-dependent function.

Suppose the input $\omega(t)$ is bounded. We need to prove $e(t) \to 0$.

Select a Lyapunov V and get its derivative $\dot{V}$:

$$V = e^2 + g^2 \tag{14}$$

$$\dot{V} = 2e\dot{e} + 2g\dot{g} = -2e^2 \leq 0 \; \forall e \tag{15}$$

According to Lagrange's theorem $V(t) \leq V(0)$, means that $V(t)$ is bounded.

Form (14) to have in-equation (16):

$$\begin{cases} e^2 \leq V \\ g^2 \leq V \end{cases} \Rightarrow \begin{cases} |e| \leq \sqrt{V} \leq \sqrt{V(0)} \\ |g| \leq \sqrt{V} \leq \sqrt{V(0)} \end{cases} \tag{16}$$

So $e(t)$ and $g(t)$ are also bounded.

However, at this point, it is not possible to conclude that $e(t)$ is asymptotically stable because $\dot{V}(t)$ is not negative but only negative semi-definitely. Furthermore, since (13) is a non-autonomous system, it is impossible to apply LaSalle's invariant set principle but to use Barbalat's lemma in this case. Accordingly, we need to calculate the second derivative of $V(t)$ and consider its boundedness.

From (15):

$$\begin{aligned} \ddot{V}(t) &= -4e\dot{e} \\ &= -4e[-e + g\omega] \end{aligned} \tag{17}$$

Since $e(t)$, $g(t)$ and $\omega(t)$ are bounded so $\ddot{V}(t)$ is bounded, which means that $\dot{V}(t)$ is uniformly continuous. According to Barbalat's lemma:

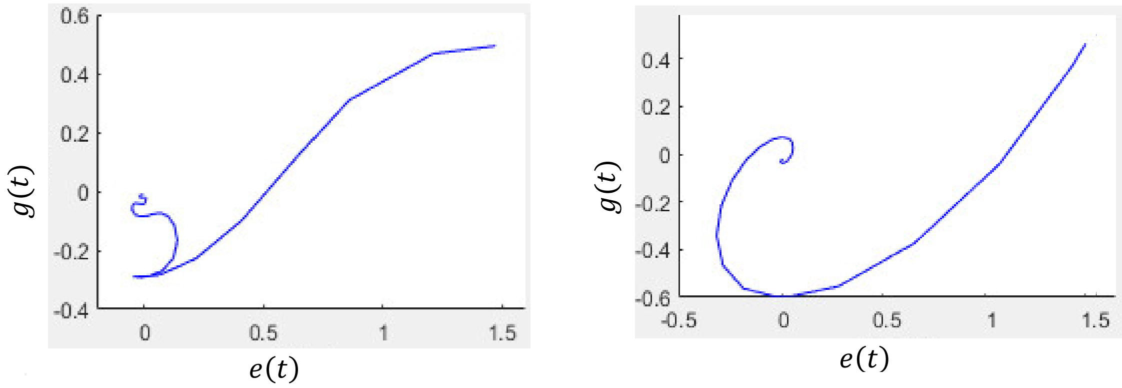
$$\begin{aligned} &\lim_{t \to \infty} \dot{V}(t) = 0 \\ \Rightarrow &\lim_{t \to \infty} e(t) = 0 \end{aligned} \tag{18}$$

Therefore $e(t)$ is asymptotically stable.

The results of system simulation (13) are shown in figure 1, showing the convergent states of the system with inputs $\sin(t)$ (figure 1a) and $1(t)$ (figure 1b).

Remarks:

- The initial states of the system are $[e(t) \;\; g(t)]^T = [1.5 \;\; 0.5]^T$.
- Simulation is performed after $20s$ the states of the system converge to 0.
- With different inputs, system (13) is asymptotically stable.
- The simulation results are consistent with the theoretical basis.



a) Convergent states with $\omega(t) = sin(t)$. *b) Convergent states with $\omega(t) = 1(t)$.*

***Figure 1.** The convergent states of the system with different inputs.*

3.2. Analysis of asymptotic stability of a drive control system

3.2.1. Theoretical analysis

Consider a drive control system with the dynamical equation described in (19).

$$J\ddot{\theta}(t) = -f(\theta, t) + bu(t) \tag{19}$$

Tracking angle error and its derivative:

$$\begin{cases} e(t) = \theta_d(t) - \theta(t) \\ \dot{e}(t) = \dot{\theta}_d(t) - \dot{\theta}(t) \end{cases} \Rightarrow \begin{cases} \dot{\theta}(t) = \dot{\theta}_d(t) - \dot{e}(t) \\ \ddot{\theta}(t) = \ddot{\theta}_d(t) - \ddot{e}(t) \end{cases} \tag{20}$$

Where: θ is the actual angle.

$f(\theta, \dot{\theta}, t) = a\dot{\theta}$ is the term depends on speed angle with vicious coefficient a.

u is the control signal.

b is the control matrix.

J is the inertial moment of the system.

θ_d is the desired angle.

Reform (19) under the tracking error equation by substituting (20) into (19):

$$J(\ddot{\theta}_d - \ddot{e}) = -a(\dot{\theta}_d - \dot{e}) + bu \tag{21}$$

$$J\ddot{e} + a\dot{e} + bu - (J\ddot{\theta}_d + a\dot{\theta}_d) = 0 \tag{22}$$

Set:

$$x_1 = e(t); x_2 = \dot{e}(t) \tag{23}$$

Without loss of generality, for simplicity in mathematical representation, consider $J = 1\ kgm^2$ then the system (22), (23) is reformed as the state equation (24):

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -ax_2 - bu + (\ddot{\theta}_d + a\dot{\theta}_d) \end{cases} \tag{24}$$

Choose the autonomous system (25) to be the sliding surface for system (24):

$$S = \lambda x_1 + x_2 = 0, \lambda > 0 \tag{25}$$

Suppose we have synthesized the controller u such that the derivative of S has the form (26):

$$\dot{S} = -\varepsilon \text{sign}(S) - kS;\ \varepsilon, k > 0 \tag{26}$$

To analyze the asymptotical stability of the system, select the Lyapunov V.

$$V = \frac{1}{2}S^2 \tag{27}$$

$$\dot{V} = S\dot{S} \tag{28}$$

Substitute (26) into (28):

$$\dot{V} = -\varepsilon|S| - k|S|^2 \leq 0\ \forall S \tag{29}$$

From (29), according to Lagrange's theorem, $V(t)$ is a decreasing function (monotonical decreasing) with $\forall S \neq 0$, which means that when comparing $V(t)$ at the initial time $t = 0$ and later time $t > 0$, there's always $V(t) \leq V(0)$, means that $V(t)$ is bounded. Since $V(t)$ is bounded, so according to (12), S is also bounded.

Because $\dot{V}(t)$ is semi-negative, so it is impossible to conclude $|S| \to 0$. To apply Barbalat's lemma, we need to calculate the second derivative of $V(t)$ and make sure that $\ddot{V}(t)$ is bounded to be possible to conclude $\dot{V}(t)$ is uniformly continuous.

Indeed, from (29), calculate the second derivative of $V(t)$:

$$\ddot{V} = -\varepsilon|\dot{S}| - 2kS\dot{S} \tag{30}$$

Substitute (26) into (30):

$$\ddot{V} = -\varepsilon|\varepsilon \text{sign}(S) + kS| + 2\varepsilon k|S| + 2k^2S^2 \tag{31}$$

Formula (31) shows that $\ddot{V}$ is the function of bounded terms S, therefore $\ddot{V}$ is bounded and deduces $\dot{V}$ is uniformly continuous.

Barbalat's lemma allows us to conclude:

$$\dot{V} = -\varepsilon|S| - k|S|^2 \to 0 \tag{32}$$

Therefore $|S| \to 0$.

Thus, if the control law ensures the derivative of S has the form (26), the control system is asymptotically stable. Furthermore, if the control law guarantees for $S \to 0$ in finite time, then at the time t_0 when the state of the system falls on the sliding surface, the equation $S = 0$ exists.

From (23) to have:

$$\begin{gathered} S = \lambda x_1 + x_2 = 0\ \Rightarrow \lambda x_1 = -x_2 \Leftrightarrow \lambda x_1 = -\dot{x}_1 \\ \Rightarrow \ln|x_1| = -\frac{1}{\lambda}t + C, \qquad C = const \end{gathered} \tag{33}$$

Suppose the system states at the time on the sliding surface: $x_{1s}|_{t=t_0} = x_{1s}(t_0)$, one has:

$$C = \ln|x_{1s}(t_0)\,| + \frac{1}{\lambda}t_0 \tag{34}$$

Substitute (34) into (33) to have:

$$\ln|x_1| = -\frac{1}{\lambda}t + \ln|x_{1s}(t_0)| \tag{35}$$

$$x_1 = x_{1s}(t_0)e^{-\frac{1}{\lambda}(t-t_0)} \tag{36}$$

$$x_2 = -\frac{x_{1s}(t_0)}{\lambda} e^{-\frac{1}{\lambda}(t-t_0)}$$

Since $\lambda > 0$ so, from (36) shows that $x_1, x_2 \to 0$ when $t \to \infty$.

3.2.2. A simulation example

Suppose the parameters of the system (24) as (37) [9]:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -25x_2 - 133u_v + \left(\ddot{\theta}_d + 25\dot{\theta}_d\right) \end{cases} \quad (37)$$

Assuming initial angular position and angular speed:

$$\theta_d(0) = -0.5\, rad;\ \dot{\theta}_d(0) = -0.5\, rad/s \quad (38)$$

With the step input desired angle $\theta_d(t) = 1(t)$, the initial system states are:

$$[x_1\ \ x_2]^T = [1.5 \quad 0.5]^T \quad (39)$$

Select the sliding surface (40):

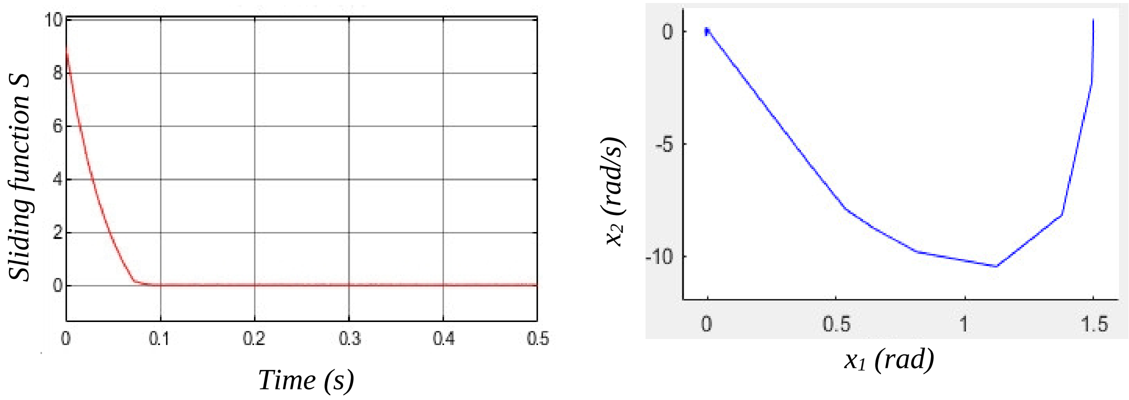
$$S = 15x_1 + x_2 \quad (40)$$

The approaching law to sliding surface $S = 0$:

$$\dot{S} = -70\text{sgn}(15x_1 + x_2) - 20(15x_1 + x_2) \quad (41)$$

The controller is synthesized according to [9]:

$$u = \frac{1}{133}[20(15x_1 + x_2) + 70\text{sgn}(15x_1 + x_2) - 10x_2] \quad (42)$$



a) Convergence of S to 0. *b) Convergence of the system states to 0.*

Figure 2. *Sliding surface and convergent states of the tracking drive system.*

Remarks:

- Figure 2 shows that in the case of a system using control laws, the sliding surface $S = 0$ (an intermediate autonomous system in the control system) always exists, and the system states converge to 0.

- The two examples above are typical cases illustrating the application of Barbalat's lemma when proving the asymptotic stability of a dynamical system according to the Lyapunov stability principle.

4. CONCLUSIONS

The asymptotic stability of a non-autonomous dynamical system is considered based on the Lyapunov stability theory. The bounded property of a form of the Lyapunov function has been confirmed by Lagrange's theorem, which serves as the basis for the application of Barbalat's lemma, ensuring the asymptotic stability of the system. The article has explained and clarified the application of Barbalat's lemma and Lagrange's theorem; through 2 typical examples to confirm the solid mathematical basis in analyzing the asymptotic stability of a dynamical system. Of course, the main difficulty in using Barbalat's lemma to examine the stability of dynamical systems is choosing the appropriate Lyapunov function.

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TÓM TẮT

Tính ổn định tiệm cận của hệ thống động với bổ đề Barbalat và hàm Lyapnunov

Bài báo diễn giải, kết hợp ứng dụng bổ đề Barbalat, hàm Lyapunov và định lý Lagrange nhằm đảm bảo toán học vững chắc trong việc phân tích tính ổn định tiệm cận của một hệ thống động. Kết quả nghiên cứu được minh họa và mô phỏng bằng một số ví dụ trực quan, cho cả hệ thống động học không điều khiển và hệ thống động lực học có điều khiển.

Từ khoá: Bổ đề Barbalat; Hàm Lyapunov; Định lý Lagrange; Ổn định tiệm cận; Hệ không tự trị.